(1) Match each function to its graph, justifying your choices.

(I) $z = \sin(x - y) \iff \langle a \rangle$ (f) Note the periodicity as well as the fact that $z$ is constant whenever $x - y$ is constant.

(II) $z = \sin(xy) \iff \langle b \rangle$ (e) Again we have periodicity; and when $xy$ is constant, that is, when $(x, y)$ is on a hyperbola, then $z$ is constant.

(III) $z = e^x \cos(y) \iff \langle c \rangle$ (d) This is clearly a cosine wave in the $y$ direction whose amplitude grows exponentially in the $x$ direction.

(IV) $z = \sin(x) - \sin(y) \iff \langle d \rangle$ Either by elimination of the periodic graphs, or by noting that the graph is a (shifted) sine wave whenever either $x$ or $y$ is constant.

(V) $z = (1 - x^2)(1 - y^2) \iff \langle e \rangle$ As $x$ and $y$ grow, so does $z$, and there’s symmetry between positive and negative $x$ and $y$ because of the squares.

(VI) $z = \frac{x - y}{1 + x^2 + y^2} \iff \langle f \rangle$ Note the periodicity as well as the fact that $z$ goes to 0 as either $x$ or $y$ or both move away from the origin, since the denominator (quadratic) grows faster than the numerator (linear).

(2) Given two linearly independent vectors $\vec{v} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, we wish to find all vectors $\vec{v} = \langle x, y, z \rangle$ such that $\vec{v}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

(a) Using the dot product, express the condition “$\vec{v}$ is orthogonal to each of $\vec{a}$ and $\vec{b}$” as a system of two linear equations in three unknowns.

We want $\vec{a} \cdot \vec{v} = \vec{b} \cdot \vec{v} = 0$, and so,

$$a_1 x + a_2 y + a_3 z = 0$$
$$b_1 x + b_2 y + b_3 z = 0$$

(b) Find the general solution to the system of linear equations above. What is the dimension of the solution space? Does that match your geometric intuition? You may assume, for simplicity, that $a_1 \neq 0$ and $b_2 \neq a_2 b_1 / a_1$.

Using row reduction, we have:

$$\begin{align*}
    a_1 x + a_2 y + a_3 z &= 0 \\
    b_1 x + b_2 y + b_3 z &= 0
\end{align*}$$

$$\begin{align*}
    \rightarrow a_1 x + a_2 y + a_3 z &= 0 \\
    y + \frac{a_1 b_1 - a_2 b_2}{a_1 b_2 - a_2 b_1} z &= 0 \\
    \rightarrow a_1 x + a_2 y + a_3 z &= 0 \\
    y + \frac{a_1 b_1 - a_2 b_2}{a_1 b_2 - a_2 b_1} z &= 0
\end{align*}$$

$$\begin{align*}
    x + \frac{a_3 b_2 - a_2 b_3}{a_1 b_2 - a_2 b_1} z &= 0 \\
    y + \frac{a_3 b_2 - a_2 b_3}{a_1 b_2 - a_2 b_1} z &= 0
\end{align*}$$

$$\begin{align*}
    x &= \frac{a_3 b_2 - a_2 b_3}{a_1 b_2 - a_2 b_1} t \\
    y &= \frac{a_3 b_2 - a_2 b_3}{a_1 b_2 - a_2 b_1} t \\
    z &= \frac{a_3 b_2 - a_2 b_3}{a_1 b_2 - a_2 b_1} t
\end{align*}$$
The dimension of the solution space, which equals the number of free variables, is 1; this makes sense, since in \( \mathbb{R}^3 \), there is one direction, or a one-dimensional subspace, perpendicular to two given vectors.

(c) Is there a choice of the free variable in the general solution that makes \( \vec{v} = \vec{a} \times \vec{b} \)? Conclude that \( \vec{a} \times \vec{b} \) is indeed orthogonal to both \( \vec{a} \) and \( \vec{b} \).

If we choose \( t = a_1b_2 - a_2b_1 \), we get \( x = a_2b_3 - a_3b_2, \ y = a_3b_1 - a_1b_3, \) and \( z = a_1b_2 - a_2b_1 \). Namely, we get the components of the vector \( \vec{a} \times \vec{b} \). Since this is one of many solutions to the system of linear equations \( \vec{a} \cdot \vec{v} = \vec{b} \cdot \vec{v} = 0 \), we conclude that \( \vec{a} \times \vec{b} \) is orthogonal to both \( \vec{a} \) and \( \vec{b} \).

(3) Find the equation of the plane that contains the points \( (0, 0, 0) \), \( (1, 1, -1) \) and \( (1, 2, 3) \).

To find the equation of a plane, we need a point (we have three!) and a normal vector. By subtracting point coordinates, we see that the vectors \( \vec{x} = \langle -1, 1, -2 \rangle \) and \( \vec{y} = \langle -1, 1, -2 \rangle \) are parallel to our plane.

Hence, their cross product will be perpendicular to the plane: \( \vec{n} = \langle 1, 1, -1 \rangle \times \langle 1, 2, 3 \rangle = \langle -5, -4, 1 \rangle \).

Therefore, the equation of the plane is \( 5(x - 0) - 4(y - 0) + (z - 0) = 0 \), or \( 5x - 4y + z = 0 \).

(4) Find the parametric equation of the straight-line intersection of the planes \( z = -x + y \) and \( -x + 3z = 2 \).

One way to do this is to solve the corresponding system of two equations in three unknowns:

\[
\begin{align*}
-x + y - z &= 0 \\
-x + 3z &= 2
\end{align*}
\]

\[
\begin{align*}
-x + y - z &= 0 \\
y - 4z &= -2
\end{align*}
\]

\[
\begin{align*}
-x + y - z &= 0 \\
x - 3z &= -2 \\
y - 4z &= -2
\end{align*}
\]

Since \( z \) is a free variable, we can set \( z = t \), and then \( x = -2 + 3t \) and \( y = -2 + 4t \). So the parametric equation we seek is \( \vec{r}(t) = \langle -2 + 3t, -2 + 4t, t \rangle \).

We could have also found two points on the intersection and used them to find the requisite parallel vector. Keep in mind that it’s possible to get two different parameterizations of the same straight line, as we’ve seen in class, since two different moving objects can trace the same path at different times.

We could have as well found a point in common on the two planes, and taken the cross product of the two normal vectors to use as the parallel vector.

(5) In the previous homework, we met 0- and 1-forms over \( \mathbb{R} \), and we paraphrased the Fundamental Theorem of Calculus as an equality of evaluated forms. Namely, if \( f \) is a 0-form, \( df \) is the corresponding 1-form, and \([a, b]\) is an interval, then the Fundamental Theorem of Calculus says that \( f \) evaluated on the boundary of \([a, b]\) equals \( df \) evaluated on \([a, b]\).

Recall that we required functions that appear in 0-forms and 1-forms be “nice” functions from \( \mathbb{R} \) to itself. Let’s make that a bit more precise here. For \( d : \{0\text{-forms}\} \rightarrow \{1\text{-forms}\} \) to make any sense, the 0-forms must be differentiable functions. Let’s assume for the sake of uniformity that the functions \( g(x) \) in the 1-forms \( g(x)dx \) are differentiable as well. Consider the 1-form \( \omega = \cos(x)dx \); you can probably find a 0-form \( \varphi = f(x) \) such that \( d\varphi = \omega \). Does every 1-form have to come from a 0-form by way of \( d \) in the same way? Prove or give a counterexample.

Suppose \( g(x)dx \) is a nice 1-form, and define \( f(x) = \int_{t=x}^{x=t} g(t)dt \). By the definition of the definite integral, we know that \( f(x) \) measures the (signed) area under the graph \( y = g(t) \) from \( t = 0 \) to \( t = x \) (hence, for each \( x \), there is a different area). And by the Fundamental Theorem of Calculus, we know that \( f'(x) = g(x) \); therefore, \( df(x) = g(x)dx \).

[Note that we could have been more precise here by specifying the number of times the functions are differentiable. For instance, if we required our 0-forms to be twice-differentiable, then we’d want the functions that make up our 1-forms to be once-differentiable.]