(1) Suppose $A$ is an $n \times n$ matrix. Recall that if $p(x)$ is a polynomial, we can substitute $A$ for $x$ (see for instance problem (30c) on homework 12R). Let $p_A(\lambda) = \det(A - \lambda I_n)$ be the polynomial of degree $n$ whose roots are the eigenvalues of $A$. It may be useful for the questions that follow to convince yourself of the fact that, if $B$ is an $n \times n$ matrix, then $\det(A - BI_n)$ is not the same as $p_A(B)$.

(a) Suppose $A$ is diagonalizable and $p(x)$ is a polynomial. Show that $p(A)$ is also diagonalizable.

(b) For $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, compute $p_A(\lambda)$, $p_A(B)$, and $p_A(A)$.

(c) Suppose $A$ is diagonalizable. Show that $p_A(A)$ is the $n \times n$ matrix whose entries all equal 0.

(2) For an $n \times n$ matrix $A$, denote as usual the entry in row $i$, column $j$ by $a_{ij}$. Define the trace of $A$ to be the sum of its diagonal entries: $\text{trace}(A) = a_{11} + a_{22} + \cdots + a_{nn}$.

(a) Show that $\text{trace}(A+B) = \text{trace}(A) + \text{trace}(B)$. Is $\text{trace}(AB) = \text{trace}(A)\text{trace}(B)$?

(b) Suppose $A$ and $B$ are $n \times n$ matrices. For $n = 2, 3$, prove that $\text{trace}(AB) = \text{trace}(BA)$. Can you prove it for a general $n$?

(c) Suppose $A$ is a diagonalizable matrix. Show that $\text{trace}(A)$ equals the sum of the $n$ eigenvalues of $A$.

(d) Extra Credit. Prove (c) for any square matrix (without looking the proof up online).

(3) Compute the eigenvalues of the following matrices by inspection, without solving $\det(A - \lambda I_n) = 0$. Justify your answers.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) $\begin{pmatrix} 1 &amp; 5 \ 3 &amp; 3 \end{pmatrix}$</td>
<td>(b) $\begin{pmatrix} 3 &amp; -1 \ 4 &amp; 8 \end{pmatrix}$</td>
<td>(c) $\begin{pmatrix} 1 &amp; 2 \ 2 &amp; 4 \end{pmatrix}$</td>
</tr>
</tbody>
</table>