CHAPTER 6

Eigentheory

1. Blah Blah

Who is a matrix, really? Complex creatures like matrices are better described in terms of what they do instead of who they are. Matrices can be associated to soles, vector subspaces, linear transformations, parallelotopes, and determinants. They can exist as abstract objects in their own rights, with sums, products, ranks, and inverses lending them structure. So if we encounter a matrix on the street, we would be reductive to assume we know it well just because we’ve read off the array of numbers that make it up. Yet, we’ve come to believe that a matrix has an essence, a true self.

In this chapter, we will tease out the self of a square matrix by way of its eigenvectors and eigenvalues. We will harness some of the eigenpower to answer questions we asked many chapters ago, and we will look forward to where the new eigentools might lead us. Eigen eigen eigen.

According to the all-knowing Google,\textsuperscript{1} the top three English translations of the German prefix ‘eigen-’ are ‘own,’ ‘inherent,’ and ‘peculiar.’ So we can expect that the eigenvectors and eigenvalues of a given matrix will tell us some fundamental facts about the true nature of that matrix. At second glance, ‘peculiar’ would be understood to mean ‘characteristic,’ as in ‘A walked with a singular swagger peculiar to noninvertible matrices.’ However, the first glance may have something to contribute, too. Eigenstuff is peculiar.

2. Eigendefinitions

Cue the trumpets. We are here.

DEFINITION 1. Suppose $A$ is an $n \times n$ matrix. The nonzero vector $\mathbf{v} \in \mathbb{R}^n$ is called an eigenvector of $A$ with eigenvalue the scalar $\lambda \in \mathbb{R}$ if $A\mathbf{v} = \lambda \mathbf{v}$.

Anything that can be put in a nutshell belongs there,\textsuperscript{2} so we content ourselves with a selection of preliminary impressions for now. Our understanding will expand with time.

We cobble together relevant bits of info from the previous two chapters to illustrate the process of computing eigenvectors/eigenvalues, the payoff that such a computation yields, and the way in which eigentheory makes use of so many of the topics we’ve covered.

EXAMPLE 2. We have a long history with the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, most recently in Example 49 of Chapter 4. Let’s reconstruct our adventure by highlighting the flow of our thought process from Chapter 4, adding a conclusion from Chapter 5 to it.

Suppose $L : \mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2$ is the linear transformation with matrix representation $A$. We wish to find a basis $B$ such that the matrix representation of $L$ in the coordinates of $B$ is diagonal.

We love diagonal matrices because they reveal geometric secrets of linear transformations. Additionally, matrix computations can be much easier and more efficient with diagonal matrices.

\textsuperscript{1}The way Google decides which hits appear on the first page is a whole lot of linear algebra!

\textsuperscript{2}Versions of the quip attributed to Sydney J. Harris & H. L. Mencken & Warren Goldfarb & Hilary Putnam.

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We want to find a basis $B = \{ \vec{v}_1, \vec{v}_2 \}$ such that

$$L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

and $\lambda_1 \vec{v}_1$ is a basis vector, since we are solving a homogeneous sole and we expect to get a nontrivial answer.

By Observation 48 of Chapter 4, this means

$$A\vec{v} = \lambda \vec{v} \iff A\vec{v} = \lambda \vec{v}_{\vec{0}} \iff A\vec{v} - \lambda \vec{v} = \vec{0} \iff (A - \lambda I)\vec{v} = \vec{0} \iff \vec{v} \in \mathcal{N}(A - \lambda I).$$

Therefore, the two pairs $\lambda_1, \vec{v}_1$ and $\lambda_2, \vec{v}_2$ satisfy the nonlinear system of equations $A\vec{v} = \lambda \vec{v}$.

Our plan to solve for $\lambda$ and $\vec{v}$ in $A\vec{v} = \lambda \vec{v}$ has therefore been split into two steps. First, we solve $\det(A - \lambda I) = 0$. Second, for each value of $\lambda$ we find, we solve the homogeneous sole $(A - \lambda I)\vec{v} = \vec{0}$.

For the first step, $0 = \det \left( \begin{array}{cc} 1 - \lambda & 2 \\ 2 & 1 - \lambda \end{array} \right) = (1 - \lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) \implies \lambda = 3, \lambda = -1$.

For the second step, with $\lambda_1 = 3$, we solve $(A - \lambda I_2)\vec{v} = \vec{0}$.

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}^4 \implies \vec{v} = \begin{pmatrix} s \\ s \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \end{pmatrix} (s \neq 0$ since we’re looking for basis vectors). We can pick $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Similarly, with $\lambda_2 = -1$, we solve $(A - \lambda_2 I_2)\vec{v} = \vec{0}$.

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \implies \vec{v} = \begin{pmatrix} -s \\ s \end{pmatrix} = s \begin{pmatrix} -1 \\ 1 \end{pmatrix} (s \neq 0).$ One can pick $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Therefore, $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$, and the matrix representation of $L$ in the coordinates of $B$ is $D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$.

We can verify that $D = Q^{-1}AQ$ or $A = QDQ^{-1}$, where $Q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is the change-of-basis matrix from $B$ to standard.

Note how this sheds new light on the origins of $\vec{v}_1$ and $\vec{v}_2$ in Example 49 of Chapter 4.

What does $D$ do for us? For one thing, it explains how $L$ acts geometrically: if $\vec{v} \in \mathbb{R}^n$, $L(\vec{v})$ triples the component of $\vec{v}$ in the direction of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and flips the component of $\vec{v}$ in the direction of $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. For another, we can find high powers of $A$ more quickly and elegantly with the help of $D$: since $A = QDQ^{-1}$, then $A^4 = (QDQ^{-1})(QDQ^{-1})(QDQ^{-1})(QDQ^{-1}) = QD(Q^{-1}Q)(Q^{-1}Q)(Q^{-1}Q)(Q^{-1}Q) = QD^4Q^{-1}$; and $D^4$ is much easier to compute than $A^4$. If that doesn’t convince you, try computing $A^{17}$.

To find the eigenvalues and eigenvectors of any given matrix, we generalize the insight from Example 2.

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3Since $\vec{v}_1$ and $\vec{v}_2$ form a basis, they must be nonzero. Since we additionally have $A\vec{v}_1 = \lambda_1 \vec{v}_1$ and $A\vec{v}_2 = \lambda_2 \vec{v}_2$, we now know to call $\vec{v}_1$ and $\vec{v}_2$ eigenvectors of $A$ with eigenvalues $\lambda_1$ and $\lambda_2$, respectively.

4We always get a free variable in eigenvector computations, since we are solving a homogeneous sole and we expect to get a nontrivial answer.
Observation 3. Suppose $A$ is an $n \times n$ matrix. Then

(1) $\lambda$ is an eigenvalue of $A$ if and only if $\det(A - \lambda I_n) = 0$, and

(2) $\tilde{v}$ is an eigenvector of $A$ with eigenvalue $\lambda$ if and only if $\tilde{v}$ is a nonzero vector in $\mathcal{N}(A - \lambda I_n)$.

Proof. Suppose $\tilde{v}$ is an eigenvector of $A$ with eigenvalue $\lambda$. This means $A\tilde{v} = \lambda \tilde{v} \iff A\tilde{v} - \lambda \tilde{v} = \tilde{0}_n \iff A\tilde{v} - \lambda I_n \tilde{v} = \tilde{0}_n \iff (A - \lambda I_n)\tilde{v} = \tilde{0}_n \iff \tilde{v} \in \mathcal{N}(A - \lambda I_n)$ and $\tilde{v} \neq \tilde{0}_n$; this proves (2). And we know $\mathcal{N}(A - \lambda I_n)$ contains a nonzero vector precisely when $A - \lambda I_n$ is not invertible, which happens if and only if $\det(A - \lambda I_n) = 0$, proving (1). \hfill \Box

In this way, Observation 3 gives us a direct way to find eigenvectors and eigenvalues of a given $n \times n$ matrix $A$ in two easy steps.

(1) Solve for all $\lambda$ in the equation $\det(A - \lambda I_n) = 0$.

(2) For each $\lambda$ computed in (1), find all nonzero solutions to the homogeneous system $\left( A - \lambda I_n \right) \tilde{0}_n$.\footnote{You may hear the set of all solutions to $\left( A - \lambda I_n \right) \tilde{0}_n$, or equivalently, $\mathcal{N}(A - \lambda I_n)$, called the eigenspace of $\lambda$, and denoted $E_\lambda$. Since we know the nullspace to be a vector subspace, $E_\lambda$ is a vector subspace of $\mathbb{R}^n$. It contains all the eigenvectors of $A$ with eigenvalue $\lambda$ as well as the zero vector. We need the zero vector to make a subspace, so we allow its inclusion among the eigenvectors, although it is not one of them.}

3. Towards Diagonalization

With a trusty algorithm in hand, we begin to explore nuances of eigenfeatures. For instance, the more eigenvectors we have, the more we can simplify a matrix.

Observation 4. Suppose $A$ is an $n \times n$ matrix and $B = \{\tilde{v}_1, \ldots, \tilde{v}_n\}$ is a basis for $\mathbb{R}^n$. Let $L_A : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation whose matrix representation is $A$, and let $D$ be the matrix representation of $L$ in the coordinates of $B$. Then $\tilde{v}_i$ is an eigenvector of $A$ with eigenvalue $\lambda_i$ if and only if the $i$th column of $D$ equals $\lambda_i \tilde{v}_i$. In particular, $D = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$ if and only if $\tilde{v}_i$ is an eigenvector of $A$ with eigenvalue $\lambda_i$ for all $1 \leq i \leq n$.

Proof. This simply paraphrases Observation 48 of Chapter 4 using Definition 1. By Observation 44 of Chapter 4, if the $i$th column of $D$ equals $\lambda_i \tilde{v}_i$, that’s the same as saying $L(\tilde{v}_i) = \begin{pmatrix} \lambda_i \tilde{v}_i \end{pmatrix}_B = \lambda_i \tilde{v}_i$, or equivalently, $A\tilde{v}_i = \lambda_i \tilde{v}_i$. \hfill \Box

Note that, in Observation 4, we have $A = QDQ^{-1}$, where $Q$ is the change-of-basis matrix from $\mathbb{R}^n_B$ to $\mathbb{R}^n$ (see Section 7.4 of Chapter 4 for more such memories). The potential of $D$ being diagonal inspires us to add a definition to our repertoire.

Definition 5. The $n \times n$ matrix $A$ is called diagonalizable if there exists an invertible matrix $Q$ and a diagonal matrix $D$ such that $A = QDQ^{-1}$. The process of finding $Q$ and $D$ is called diagonalizing $A$.

Diagonalizability is a lovely concept. For one thing, factoring a matrix in this manner, $A = QDQ^{-1}$, is going to make certain computations easier and insights deeper. For another, the formula is a meeting place for many of the fundamental ideas we’ve encountered to far, from invertibility to linear transformations, and from bases to matrix representations.

According to Observation 4, to diagonalize a matrix, we must find a basis of $\mathbb{R}^n$ consisting of eigenvectors. We have an uncontrollable urge to diagonalize every square matrix we see!

3.1. Two examples to illustrate the friendliness of eigencomputations. Observation 3 should allow us to find eigenvectors and eigenvalues to our heart’s content, and so, to diagonalize matrices.
Example 6. We have a hammer, so we go looking for a nail, which we find in the person of \( A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix} \).

With our safety goggles on, we first solve for \( \lambda \) in \( \det(A - \lambda I_3) = 0 \).

\[
\det(A - \lambda I_3) = \det\begin{pmatrix} 1-\lambda & 2 & -1 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{pmatrix}
= (1-\lambda) \det\begin{pmatrix} -\lambda & 1 \\ -4 & 5-\lambda \end{pmatrix} - 2 \det\begin{pmatrix} 1 & 1 \\ 4 & 5-\lambda \end{pmatrix} - 1 \det\begin{pmatrix} 1 & -\lambda \\ 4 & -4 \end{pmatrix}
= (1-\lambda)(-5\lambda + \lambda^2 + 4) - 2(5 - \lambda - 4) - 1(-4 + 4\lambda)
= -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = -(\lambda - 1)(\lambda - 2)(\lambda - 3)
\]

The eigenvalues \( \lambda_1 = 1, \lambda_2 = 2, \) and \( \lambda_3 = 3 \). For each \( \lambda_i \), we next find all nontrivial solutions to the homogeneous sole \(( A - \lambda_i I_3 ) \mathbf{0}_3 \).

\[
\lambda_1 = 1: \quad ( A - 1 I_3 \mathbf{0}_3 ) \rightarrow \begin{pmatrix} 0 & 2 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 4 & -4 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 -1 & 0 \\ 4 & -4 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

So the eigenvectors with eigenvalue \( \lambda_1 = 1 \) are \( \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ t \end{pmatrix} = t \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix} \), with \( t \neq 0 \).

\[
\lambda_2 = 2: \quad ( A - 2 I_3 \mathbf{0}_3 ) \rightarrow \begin{pmatrix} -1 & 2 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 4 & -4 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & -4 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

So the eigenvectors associated to the eigenvalue \( \lambda_2 = 2 \) are \( \begin{pmatrix} -\frac{1}{4} \\ \frac{1}{4} \\ t \end{pmatrix} = t \begin{pmatrix} -\frac{1}{4} \\ \frac{1}{4} \\ 1 \end{pmatrix} \), with \( t \neq 0 \).

\[
\lambda_3 = 3: \quad ( A - 3 I_3 \mathbf{0}_3 ) \rightarrow \begin{pmatrix} 2 & 2 & -1 & 0 \\ 1 & -3 & 1 & 0 \\ 4 & -4 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -3 & 1 & 0 \\ 0 & -4 & 2 & 0 \\ 4 & -4 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -3 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

So the eigenvectors associated to the eigenvalue \( \lambda_3 = 3 \) are \( \begin{pmatrix} -\frac{1}{4} \\ \frac{1}{4} \\ t \end{pmatrix} = t \begin{pmatrix} -\frac{1}{4} \\ \frac{1}{4} \\ 1 \end{pmatrix} \), with \( t \neq 0 \).

To diagonalize \( A \), we choose one eigenvector for each eigenvalue.\(^6\) For instance, we’ll choose the ones corresponding to \( t = 1 \):

\[
\lambda_1 = 1, \mathbf{v}_1 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}; \quad \lambda_2 = 2, \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{4} \\ 1 \end{pmatrix}; \quad \lambda_3 = 3, \mathbf{v}_3 = \begin{pmatrix} -\frac{1}{4} \\ \frac{1}{4} \\ 1 \end{pmatrix}
\]

\(^6\)While there are infinitely many eigenvectors, we are limited by our interest to pick linearly independent ones.
We can use our favorite method to check that $v_1, v_2, v_3$ form a basis\(^7\) for $\mathbb{R}^3$. Then we apply Observation 4 to diagonalize $A$.

$$
\begin{pmatrix}
1 & 2 & -1 \\
1 & 0 & 1 \\
4 & -4 & 5
\end{pmatrix} = 
\begin{pmatrix}
-1/2 & -1/2 & -1/4 \\
1/4 & 1/4 & 1/4 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{pmatrix}
\begin{pmatrix}
-1/2 & -1/2 & -1/4 \\
1/4 & 1/4 & 1/4 \\
1 & 1 & 1
\end{pmatrix}^{-1}
$$

We could have chosen a different ordering of the basis, as long as we ordered the corresponding eigenvalues in the same way.

$$
\begin{pmatrix}
1 & 2 & -1 \\
1 & 0 & 1 \\
4 & -4 & 5
\end{pmatrix} = 
\begin{pmatrix}
-1/4 & -1/2 & -1/2 \\
1/4 & 1/4 & 1/4 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
3 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{pmatrix}
\begin{pmatrix}
-1/4 & -1/2 & -1/2 \\
1/4 & 1/4 & 1/4 \\
1 & 1 & 1
\end{pmatrix}^{-1}
$$

**Example 7.** After attending a workshop at Maker Faire, you decide to try making your own nails. For instance, your favorite basis for $\mathbb{R}^2$ may be $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$, and you want these two vectors to be eigenvectors of a $2 \times 2$ matrix $A$ with eigenvalues 4 and 7, respectively. So you reverse engineer $A$ by applying Observation 4.

$$
A = \begin{pmatrix}
1 & 2 \\
3 & 5
\end{pmatrix}
\begin{pmatrix}
4 & 0 \\
0 & 7
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
3 & 5
\end{pmatrix}^{-1} = \begin{pmatrix}
1 & 2 \\
3 & 5
\end{pmatrix}
\begin{pmatrix}
4 & 0 \\
0 & 7
\end{pmatrix}
\begin{pmatrix}
-5 & 2 \\
3 & -1
\end{pmatrix} = \begin{pmatrix}
22 & -9 \\
30 & -11
\end{pmatrix}
$$

It’s straightforward to verify\(^8\) that $A$ does indeed have eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ with eigenvalues 4 and 7, respectively.

### 3.2. Two uncooperative examples to focus our resolve.

Had we lacked ambition and curiosity, we could wrap up this chapter now—we know how to algorithmically find eigenvectors and eigenvalues and diagonalize matrices. But eigentheory has a lot more elegance and insight to offer outside of this algorithm.\(^9\)

**Example 8.** The matrix from Example 2 has a relative, $M = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, that lives off the grid in Lamy, New Mexico. On a visit there, we start a conversation about the right basis, and before long, we’re computing eigenvectors and eigenvalues. In particular, we want to find a basis $B = \{ \tilde{v}_1, \tilde{v}_2 \}$ for $\mathbb{R}^2$ with the following property: if $L_M : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the linear transformation whose matrix representation is $M$, then we want the matrix representation of $L$ in the coordinates of $B$ to be diagonal: $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. In other words, we want to find a basis for $\mathbb{R}^2$ that consists of eigenvectors of $M$.

Applying Observation 3, first we solve for $\lambda$ in $\det(M - \lambda I_2) = 0$: $\det\left( \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = (1 - \lambda)^2 = 0 \iff \lambda = 1$. Therefore, $\lambda = 1$ is the only eigenvalue of $M$.\(^10\) To compute the associated eigenvectors, we find all nontrivial solutions of the homogeneous sole $\begin{pmatrix} M - I_2 & \tilde{0}_2 \end{pmatrix}$, which $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \implies \tilde{v} = \begin{pmatrix} s \\ 0 \end{pmatrix} = s \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with $s \neq 0$. Therefore, the eigenvectors of $M$ with eigenvalue $\lambda = 1$\(^11\) are nonzero scalar multiples of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

No two eigenvectors of $M$ can be linearly independent, since they are all scalar multiples of each other. As such, no two eigenvectors of $M$ can form a basis for $\mathbb{R}^2$. Since we cannot find such a basis $B = \{ \tilde{v}_1, \tilde{v}_2 \}$, we

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\(^7\)Later in this chapter, we will develop a more eigencentric approach for checking which matrices we have an eigenbasis of $\mathbb{R}^n$.

\(^8\)\(\det \begin{pmatrix}
22 & -9 \\
30 & -11
\end{pmatrix} = (22 - \lambda)(-11 - \lambda) + 270 = \lambda^2 - 11\lambda + 28 = (\lambda - 4)(\lambda - 7) = 0\) et cetera and so forth.

\(^9\)Also, the algorithm has a built-in weakness that’s as infuriating as it is poetic: if $A$ is an $n \times n$ matrix, then $\det(A - \lambda I_n)$ is a polynomial of degree $n$. If $n = 2$, we can solve $\det(A - \lambda I_2) = 0$ using the quadratic formula. If $n = 3$, there’s a cubic formula, and if $n = 4$, there’s a quartic formula. However, if $n \geq 5$, there does not exist a nice general formula to find the roots. And it’s not that a formula hasn’t been found—one can prove that a formula does not exist! There are plenty of numerical methods that approximate the roots to high levels of accuracy, exact answers are, in general, elusive, and have been since around 1927.

\(^10\)In this calculation, we have used the fact that $\lambda = 1$ appears twice as a root of the equation $(1 - \lambda)^2 = 0$. As we’ll soon see, this is a significant observation.

\(^11\)… and so, the only eigenvectors of $M$…
cannot find a change-of-basis matrix \( Q = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 \end{pmatrix} \) such that \( Q^{-1} M Q = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \) is a diagonal matrix by Observation 4. Therefore, \( M \) is not a diagonalizable matrix.

As a result of Example 8, we resolve to determine exactly when an \( n \times n \) matrix has enough eigenvectors to form a basis for \( \mathbb{R}^n \), i.e. when such a matrix is diagonalizable.

**Example 9.** Down the way from the matrix we saw in Example 8, we meet another matrix, \( C = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \), who explains to us that we’re in a matrix commune of like-minded individuals. Since we are fond of such collectives, we chat some more with \( C \) to learn its peculiarities. For instance, \( \det(C - \lambda I_2) = \det \begin{pmatrix} 1 - \lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - \lambda + 1 = 0 \implies \lambda = \frac{1+i\sqrt{3}}{2} = \frac{1+i\sqrt{3}}{2} = \frac{1+i\sqrt{3}}{2} \), where \( i = \sqrt{-1} \). Exciting! \( C \) has complex eigenvalues \( \lambda_1 = \frac{1+i\sqrt{3}}{2} \) and \( \lambda_2 = \frac{1-i\sqrt{3}}{2} \).

Since this is our first foray into complex arithmetic, let’s make a couple of preemptive observations that will simplify the computations to follow:

- \[ 1 - \lambda_1 = 1 - \frac{1+i\sqrt{3}}{2} = \frac{1-i\sqrt{3}}{2} = \lambda_2; \text{ similarly, } 1 - \lambda_2 = \lambda_1, \]
- \[ \frac{1}{\lambda_1} = \frac{2}{1+i\sqrt{3}} = \frac{2}{1+i\sqrt{3}} = \frac{2(1-i\sqrt{3})}{(1)^2 - (i\sqrt{3})^2} = \frac{1-i\sqrt{3}}{2} = \lambda_2; \text{ similarly, } \frac{1}{\lambda_2} = \lambda_1. \]

\[ \lambda_1 = \frac{1+i\sqrt{3}}{2} : \begin{pmatrix} 1 & -1 \\ 1 & \lambda_1 \end{pmatrix} \begin{pmatrix} \lambda_1 & -1 \\ 1 & -\lambda_1 \end{pmatrix} = \begin{pmatrix} \frac{1+i\sqrt{3}}{2} & \frac{1+i\sqrt{3}}{2} \\ \frac{1+i\sqrt{3}}{2} & \frac{1+i\sqrt{3}}{2} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

and so the corresponding eigenvectors are \( \begin{pmatrix} t \lambda_1 \\ t \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}, \text{ with } t \neq 0. \)

\[ \lambda_2 = \frac{1-i\sqrt{3}}{2} : \begin{pmatrix} 1 & -1 \\ 1 & \lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_2 & -1 \\ 1 & -\lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{1-i\sqrt{3}}{2} & \frac{1-i\sqrt{3}}{2} \\ \frac{1-i\sqrt{3}}{2} & \frac{1-i\sqrt{3}}{2} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

and so the corresponding eigenvectors are \( \begin{pmatrix} t \lambda_2 \\ t \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}, \text{ with } t \neq 0. \)

We diagonalize \( A \) according to Observation 4.

\[
\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1+i\sqrt{3}}{2} & \frac{1-i\sqrt{3}}{2} \\ \frac{1+i\sqrt{3}}{2} & \frac{1-i\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{1+i\sqrt{3}}{2} & 0 \\ 0 & \frac{1-i\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{1+i\sqrt{3}}{2} & 1 \\ \frac{1-i\sqrt{3}}{2} & 1 \end{pmatrix}^{-1}
\]

While \( A \) is diagonalizable, we’re a bit surprised by how such a simple-looking matrix can have such a complex diagonalization. It’s not clear that diagonalizing \( A \) has helped us better understand it.

As a result of Example 9, we resolve to investigate conditions under which matrices with real entries have real eigenvalues and eigenvectors.

### 3.3. A words about similarity.

As we’re inclined to do, we establish some new language and properties that will help us operate the rest of this chapter’s heavy machinery more safely. We ease into our preparations by examining the polynomial that started it all, \( \det(A - \lambda I_n) \), and by asking about different matrices that share the same such polynomial.

**Definition 10.** If \( A \) is an \( n \times n \) matrix, then \( \det(A - \lambda I_n) \) is called the characteristic polynomial of \( A \); and \( \det(A - \lambda I_n) = 0 \) is called the characteristic equation of \( A \).

Some matrices require a bit of elbow grease to give up their characteristic polynomials, while others give them up for a song and a mixed metaphor.

**Example 11.** In Example 6, we diagonalized \( A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix} \) by writing its as

\[
\begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix} = \begin{pmatrix} -1/2 & -1/2 & -1/4 \\ 1/2 & 1/4 & 1/4 \\ 1/2 & 1/4 & 1/4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1/2 & -1/2 & -1/4 \\ 1/2 & 1/4 & 1/4 \\ 1/2 & 1/4 & 1/4 \end{pmatrix}^{-1}
\]
To accomplish that, we had to first find the roots of the characteristic polynomial
\[ \det(A - \lambda I_3) = [-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = (1 - \lambda)(2 - \lambda)(3 - \lambda) \]

Looking at the diagonal matrix \( D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \) a little more closely, we notice that
\[ \det(D - \lambda I_3) = \det\left( \begin{pmatrix} 1 - \lambda & 0 & 0 \\ 0 & 2 - \lambda & 0 \\ 0 & 0 & 3 - \lambda \end{pmatrix} \right) = (1 - \lambda)(2 - \lambda)(3 - \lambda) = \det(A - \lambda I_3). \]

That was a scribble-free computation! More importantly, \( A \) and \( D \) have the same characteristic equation, and hence, the same eigenvalues.\(^{12}\)

**Definition 12.** Two \( n \times n \) matrices \( A \) and \( M \) are called similar if there exists an invertible matrix \( Q \) such that \( A = QMQ^{-1} \). The process of diagonalizing a given matrix is the same as the process of finding a diagonal matrix that is similar to it.\(^{13}\) Notice the symmetry in the definition: if \( A = QMQ^{-1} \), then \( M = Q^{-1}A(Q^{-1})^{-1} \). Notice also that the condition \( A = QMQ^{-1} \) has appeared naturally in the conversation about matrix representations.\(^{14}\)

As we saw in Example 11, the similar matrices \( A \) and \( D \) have the same characteristic equation. Coincidence? Can’t be!

**Observation 13.** Suppose \( A \) and \( M \) are similar \( n \times n \) matrices; that is, \( A = QMQ^{-1} \) for some invertible matrix \( Q \). Then \( A \) and \( M \) have equal characteristic polynomials, and hence, the same eigenvalues. Moreover, \( w \) is an eigenvector of \( M \) with eigenvalue \( \lambda \) if and only if \( Qw \) is an eigenvector of \( A \) with eigenvalue \( \lambda \).

**Proof.** Suppose \( A \) and \( M \) are two \( n \times n \) matrices with \( A = QMQ^{-1} \) for some invertible \( n \times n \) matrix \( Q \). Notice that \( \lambda I_n = \lambda(QI_nQ^{-1}) = Q(\lambda I_n)Q^{-1} \), so that \( A - \lambda I_n = Q(M - \lambda I_n)Q^{-1} = Q(M - \lambda I_n)Q^{-1} \). Therefore, \( \det(A - \lambda I_n) = \det(Q(M - \lambda I_n)Q^{-1}) = \det(Q)\det(M - \lambda I_n)\det(Q^{-1}) \). Since \( \det(Q^{-1}) = 1/\det(Q) \), we conclude that \( \det(A - \lambda I_n) = \det(M - \lambda I_n) \), as desired.

We verify the second claim directly: \( Mw = \lambda w \iff QMw = \lambda Qw \iff QM(Q^{-1}Qw) = \lambda(Qw) \iff Qw = \lambda(Qw) \iff Qw = \lambda(Qw) \iff A(Qw) = \lambda(A(Qw)) \). \( \square \)

To wit, similar matrices have the same eigenvalues and related eigenvectors.

### 4. Multiplicity and Diagonalizability

So far, the only nondiagonalizable matrix that we’ve seen, \( M = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \) from Example 8, had one eigenvalue, \( \lambda = 1 \). That’s because \( \det(M - \lambda I_2) = (1 - \lambda)^2 = (\lambda - 1)(\lambda - 1) \) was a quadratic polynomial with one root, \( \lambda = 1 \), that appeared twice.

We needed two linearly independent eigenvectors to form a basis according to Observation 4. Thus, one eigenvalue alone was responsible for providing these two linearly independent eigenvectors. In other words, for \( M \) to have any hope of diagonalizability, we needed to find two linearly independent vectors in \( \mathcal{N}(M - I_2) = \mathcal{N}\left( \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \right) \). However, as we saw, \( \dim(\mathcal{N}(M - I_2)) = 1 \), so there was no more than one linearly independent eigenvector of \( \lambda = 1 \).

It was precisely this conflict—\( \lambda = 1 \) appeared twice as a root of \( \det(M - \lambda I_2) \) while contributing only one linearly independent eigenvector from \( \mathcal{N}(M - \lambda I_2) \)—that caused nondiagonalizability.\(^{15}\) We are spurred into definitional action.

**Definition 14.** Suppose \( A \) is an \( n \times n \) matrix and \( \lambda_0 \) is an eigenvalue of \( A \). The **algebraic multiplicity** of \( \lambda_0 \) equals the number of times that \( \lambda_0 \) appears as a root of the polynomial equation \( \det(A - \lambda I_n) = 0 \). The **geometric multiplicity** of \( \lambda_0 \) equals \( \dim(\mathcal{N}(A - \lambda_0 I_n)) \).

---

\(^{12}\) On the other hand, the eigenvectors of \( A \) and \( D \) are different. What are the eigenvectors of a diagonal matrix?

\(^{13}\) We continue to replace old words with new ones that mean exactly the same thing.

\(^{14}\) If \( A \) is the matrix representation of a linear transformation \( L : \mathbb{R}^n \rightarrow \mathbb{R}^n \), then \( M \) is the matrix representation of \( L \) in the coordinates of \( B \), where \( B \) is the basis made up of the columns of \( Q \).

\(^{15}\) Nondiagonalizability’ may be the longest word in this book, except for ‘anthropoclimatologically.’
The relation between multiplicities isn’t immediately evident, but we can say something about the smallest a multiplicity can be.

**Observation 15.** The algebraic and geometric multiplicity of an eigenvalue must be at least 1.

**Proof.** Suppose \( \lambda_0 \) is an eigenvalue of the matrix \( A \). If \( \lambda_0 \) has an algebraic (or geometric) multiplicity of 0, that means \( \lambda_0 \) did not appear as a root of \( \det(A - \lambda I_n) \) (or, respectively, that there are no nonzero vectors in \( N(A - \lambda I_n) \)). This is impossible, since we’re already assuming that \( \lambda_0 \) is an eigenvalue of \( A \). \( \Box \)

### 4.1. How do algebraic and geometric multiplicities compare?

That’s a great question! Laying the groundwork for an answer, we begin by applying Observation 4 to say something about partially diagonalizing a matrix based on geometric multiplicity.

**Observation 16.** Suppose \( A \) is an \( n \times n \) matrix and \( \lambda_0 \) is an eigenvalue of \( A \) with geometric multiplicity \( t \). Then \( A \) is similar to a matrix of the form

\[
M = \begin{pmatrix}
\lambda_0 I_t & U \\
0 & C
\end{pmatrix}, \text{ where } U \text{ is a } t \times (n-t) \text{ matrix and } C \text{ is an } (n-t) \times (n-t) \text{ matrix}
\]

Additionally, \( \det(A - \lambda I_n) = (\lambda_0 - \lambda)^t \det(C - \lambda I_{n-t}) \).

**Proof.** By definition of geometric multiplicity, we know that \( \dim(N(A - \lambda I_n)) = t \). Let \( \tilde{v}_1, \ldots, \tilde{v}_t \) be a basis for \( N(A - \lambda_0 I_n) \); in particular, \( \tilde{v}_1, \ldots, \tilde{v}_t \) are linearly independent eigenvectors of \( A \) with eigenvalue \( \lambda_0 \). Let \( \tilde{v}_{t+1}, \ldots, \tilde{v}_n \) be \( n-t \) vectors in \( \mathbb{R}^n \) such that \( B = \{ \tilde{v}_1, \ldots, \tilde{v}_n \} \) is a basis for \( \mathbb{R}^n \).

Let \( Q \) be the change-of-basis matrix from \( \mathbb{R}^n \) to \( \tilde{v}_1, \ldots, \tilde{v}_n \), and let \( M = Q^{-1}AQ \). In other words, if \( L_A : \mathbb{R}^n \to \mathbb{R}^n \) is the linear transformation whose matrix representation is \( A \), then \( M \) is the matrix representation of \( L \) in the coordinates of \( B \). By Observation 4, the first \( t \) columns of \( M \) are \( \lambda_0 \tilde{v}_1, \ldots, \lambda_0 \tilde{v}_t \). Therefore, \( M \) is of the desired form.

Finally, by Observation 13, we know

\[
\det(A - \lambda I_n) = \det(M - \lambda I_n) = \det \begin{pmatrix}
(\lambda_0 - \lambda) I_t & U \\
0 & C - \lambda I_{n-t}
\end{pmatrix} = (\lambda_0 - \lambda)^t \det(C - \lambda I_{n-t}). \Box
\]

We can now answer the titular question.

**Observation 17.** Suppose \( A \) is an \( n \times n \) matrix and \( \lambda_0 \) is an eigenvalue of \( A \) with geometric multiplicity \( t \) and algebraic multiplicity \( s \). Let \( C \) be as described in Observation 16. Then \( s \geq t \), with strict inequality if and only if \( \lambda_0 \) is an eigenvalue of \( C \).

**Proof.** By Observation 16, we know that \( \det(A - \lambda I_n) = (\lambda_0 - \lambda)^t \det(C - \lambda I_{n-t}) \). Since \( (\lambda_0 - \lambda)^t \) appears as a factor of \( \det(A - \lambda_0 I_n) \), the algebraic multiplicity of \( \lambda_0 \) is at least \( t \). The algebraic multiplicity of \( \lambda_0 \) will be greater than \( t \) precisely when \( (\lambda_0 - \lambda) \) is a factor of \( \det(C - \lambda I_{n-t}) \), or equivalently, when \( \lambda_0 \) is an eigenvalue of \( C \). \( \Box \)

### 4.2. When is a matrix diagonalizable?

We’ve already seen in Observation 4 that a matrix is diagonalizable precisely when it has enough linearly independent eigenvectors to form a basis for \( \mathbb{R}^n \). So the question becomes, how many linearly independent eigenvectors can we pick?

We’ll answer the question in two steps.

#### 4.2.1. Step 1.

A set of eigenvectors corresponding to different eigenvalues must be linearly independent.

**Observation 18.** Suppose \( \tilde{v}_1, \ldots, \tilde{v}_k \) are eigenvectors of the matrix \( A \) with eigenvalues \( \lambda_1, \ldots, \lambda_k \). If \( \lambda_i \neq \lambda_j \) for all \( i \neq j \), then \( \tilde{v}_1, \ldots, \tilde{v}_k \) are linearly independent.

**Proof.** Assume \( \tilde{v}_1, \ldots, \tilde{v}_k \) are linearly dependent.\(^{17}\) Then there must exist a linear combination

\[
(a_1 \tilde{v}_1 + a_2 \tilde{v}_2 + \cdots + a_k \tilde{v}_k = \tilde{0}_n)
\]

\(^{16}\)See Exercise 19 of Chapter 3 for the proof of this result, that a set of linearly independent vectors can always be expanded into a basis.

\(^{17}\)We’re attempting a proof by contradiction. If we assume the opposite of what we want to prove is true, and we reach a false conclusion, then what we assumed must have been false. Therefore, its opposite, i.e. what we want to prove, must be true.
where the coefficients \( a_1, \ldots, a_k \) are not all 0. Without loss of generality, suppose that \( a_1 \neq 0 \). Observe that if \( a_2 = \cdots = a_k = 0 \), then \( \tilde{v}_1 = 0 \); that is impossible, since \( \tilde{v}_1 \) is an eigenvector. Therefore, not all of \( a_2, \ldots, a_k \) equal 0.

Multiplying the linear combination \((\mathcal{A})\) by \( A \), we get

\[
a_1 A \tilde{v}_1 + a_2 A \tilde{v}_2 + \cdots + a_k A \tilde{v}_k = a_1 \lambda_1 \tilde{v}_1 + a_2 \lambda_2 \tilde{v}_2 + \cdots + a_k \lambda_k \tilde{v}_k = \tilde{0}_n
\]

Consider the equation we get by subtracting \( \lambda_1 (\mathcal{A}) \) from \((\mathcal{A})\).

\[
(a_1 \lambda_1 \tilde{v}_1 + a_2 \lambda_2 \tilde{v}_2 + \cdots + a_k \lambda_k \tilde{v}_k) - (a_1 \lambda_1 \tilde{v}_1 + a_2 \lambda_2 \tilde{v}_2 + \cdots + a_k \lambda_1 \tilde{v}_k) = a_2 (\lambda_2 - \lambda_1) \tilde{v}_2 + \cdots + a_k (\lambda_k - \lambda_1) \tilde{v}_k = \tilde{0}_n
\]

Note that \( \lambda_i - \lambda_1 \neq 0 \) for all \( i = 2, \ldots, k \) because of the hypothesis that all the eigenvalues are distinct. Also, we know that not all \( a_2, \ldots, a_k \) equal 0. Therefore, \( \tilde{v}_2, \ldots, \tilde{v}_k \) must be linearly dependent as well.

By repeating the same argument on \( \tilde{v}_2, \ldots, \tilde{v}_k \), we can conclude that \( \tilde{v}_3, \ldots, \tilde{v}_k \) are linearly dependent. Continuing in this manner for \( k - 1 \) iterations, we conclude that \( \tilde{v}_k \) is linearly dependent. This can happen only if \( \tilde{v}_k = \tilde{0}_n \), which is impossible since \( \tilde{v}_k \) is an eigenvector. Therefore, our initial assumption that \( \tilde{v}_1, \ldots, \tilde{v}_k \) are linearly dependent must have been false.

The payoff is immediate!

**Observation 19.** An \( n \times n \) matrix with \( n \) distinct eigenvalues must be diagonalizable.

**Proof.** By Observation 18, \( n \) distinct eigenvalues produce \( n \) linearly independent eigenvectors, which in turn must form a basis for \( \mathbb{R}^n \). By Observation 4, our matrix must be diagonalizable. \( \square \)

**Example 20.** We are entertaining bids on a linear algebraic maintenance job in our backyard, and we’ve narrowed the contractors down to the two \( 3 \times 3 \) matrices

\[
A = \begin{pmatrix}
1 & 4 & 7 \\
0 & 3 & 1 \\
0 & 0 & -5
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
11 & -2 & -2 \\
18 & -4 & -3 \\
48 & -8 & -9
\end{pmatrix}
\]

Certified diagonalizability is important to us, so we investigate further.

Since \( \det(A - \lambda I_3) = (1 - \lambda)(3 - \lambda)(-5 - \lambda) \), we conclude that the eigenvalues of \( A \) are 1, 3, and -5. In particular, \( A \) has three distinct eigenvalues, each with algebraic multiplicity 1.\(^8\) So \( A \) is diagonalizable by Observation 19. Diagonalizing \( A \) requires some more computations, but we can be assured at this stage that \( A \) is indeed diagonalizable.

On the other hand, \( \det(B - \lambda I_3) = [\text{scribble scribble scribble}] = -\lambda(\lambda + 1)^2 \). Therefore, the eigenvalue \(-1\) has algebraic multiplicity 2; if its geometric multiplicity is 1, that would leave us with at most two linearly independent eigenvectors. Then \( B \) would not be diagonalizable. So we set about to compute the eigenvectors of \( B \).

\[
\lambda = 0: \begin{pmatrix}
11 & -2 & -2 \\
18 & -4 & -3 \\
48 & -8 & -9
\end{pmatrix}
\rightarrow
\begin{pmatrix}
11 & -2 & -2 \\
0 & -8/11 & 3/11 \\
0 & 8/11 & -3/11
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & -1/4 \\
0 & 1 & -3/8 \\
0 & 0 & 0
\end{pmatrix}
\]

So the eigenvectors with eigenvalue 0 are \( \tilde{v} = \begin{pmatrix} t/4 \\ 3/8 \\ t \end{pmatrix} \) where \( t \neq 0 \). Therefore, \( \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \) (corresponding to \( t = 8 \)) forms a basis for \( \mathcal{N}(B) \). As expected, the geometric multiplicity of \( \lambda = 0 \) is 1, like its algebraic multiplicity.

\[
\lambda = -1: \begin{pmatrix}
12 & -2 & -2 \\
18 & -3 & -3 \\
48 & -8 & -8
\end{pmatrix}
\rightarrow
\begin{pmatrix}
12 & -2 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & -1/6 & -1/6 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\(^8\)Note that an eigenvalue with algebraic multiplicity 1 must have geometric multiplicity 1 as well, since the geometric multiplicity must be at least 1 but can’t be greater than the algebraic multiplicity by Observation 17.
So the eigenvectors with eigenvalue \(-1\) are \(\vec{v} = \begin{pmatrix} 1/6 + 1/6 \\ s \\ t \end{pmatrix} = s \begin{pmatrix} 1/6 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1/6 \\ 0 \\ 1 \end{pmatrix}\), where \(s\) and \(t\) are not both equal 0. Therefore, \(\begin{pmatrix} 1 \\ 6 \\ 0 \\ 0 \\ 1 \end{pmatrix}\) (corresponding to \(s = 6, t = 0\) and \(s = 0, t = 6\), respectively) form a basis for \(\mathcal{N}(B + I_3)\). The geometric multiplicity of \(\lambda = -1\) is 2, like its algebraic multiplicity.

Finally, \(\begin{pmatrix} 2 \\ 3 \\ 0 \\ 6 \\ 0 \end{pmatrix}\), \(\begin{pmatrix} 1 \\ 6 \\ 0 \end{pmatrix}\), and \(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\) form a basis\(^{19}\) for \(\mathbb{R}^3\). Therefore, by Observation 4, \(B\) is diagonalizable as well.

Choosing contractors is hard! We decide to buy a how-to book and perform the maintenance ourselves.

4.2.2. Step 2. We need \(n\) linearly independent eigenvectors to diagonalize a matrix. Example 20 suggests that for that to happen, we need each eigenvalue to contribute as many linearly independent eigenvectors (that’s its geometric multiplicity) as the number of times it appears as a root of the characteristic equation (that’s its algebraic multiplicity). While most eigenvalues have geometric and algebraic multiplicities equal to 1,\(^{20}\) we have much insight to gain from studying the pathological cases.

**Observation 21.** A matrix is diagonalizable if and only if the algebraic and geometric multiplicities of each of its eigenvalues are equal.

**Proof.** Suppose \(A\) is an \(n \times n\) matrix. Then its characteristic polynomial \(\det(A - \lambda I_n)\) is a polynomial in \(\lambda\) of degree \(n\). As such, \(\det(A - \lambda I_n) = 0\) has \(n\) roots,\(^{21}\) counting multiplicity.\(^{22}\) Suppose the distinct eigenvalues of \(A\) are \(\lambda_1, \ldots, \lambda_k\), their algebraic multiplicities are \(s_1, \ldots, s_k\), and their geometric multiplicities are \(t_1, \ldots, t_k\). Note that \(s_1 + \cdots + s_k = n\), since there are \(n\) roots total to our characteristic polynomial.

Suppose for each \(1 \leq i \leq k\), we have \(s_i = t_i\). We will denote with the subscript \((i)\) an affiliation with the eigenvalue \(\lambda_i\). Let \(\vec{v}_{(i)}^1, \ldots, \vec{v}_{(i)}^{t_i}\) be a basis for \(\mathcal{N}(A - \lambda_i I_n)\). These are the \(t_i\) linearly independent eigenvectors with eigenvalue \(\lambda_i\) guaranteed by the fact that the geometric multiplicity of \(\lambda_i\) equals \(t_i\). We wish to prove that all these eigenvectors,

\[
\vec{v}_{(1)}^1, \ldots, \vec{v}_{(1)}^{t_1}, \ldots, \vec{v}_{(k)}^1, \ldots, \vec{v}_{(k)}^{t_k},
\]

are linearly independent. Note that there are \(t_1 + \cdots + t_k = s_1 + \cdots + s_k = n\) of these eigenvectors in total. So if we prove that they are linearly independent, we will have proven that they form a basis for \(\mathbb{R}^n\).

Suppose some linear combination of these vectors equals the zero vector.

\[
a_1^{(1)} \vec{v}_1^1 + \cdots + a_{t_1}^{(1)} \vec{v}_{t_1}^1 + \cdots + a_1^{(k)} \vec{v}_1^{(k)} + \cdots + a_{t_k}^{(k)} \vec{v}_{t_k}^{(k)} = \vec{0}_n
\]

In other words, we have

\[
(\star) \quad \vec{w}^{(1)} + \cdots + \vec{w}^{(k)} = \vec{0}_n
\]

For each \(i = 1, \ldots, k\), \(\vec{w}^{(i)}\) is a linear combination of a basis of \(\mathcal{N}(A - \lambda_i I_n)\), and so \(\vec{w}^{(i)} \in \mathcal{N}(A - \lambda_i I_n)\). Since \(\mathcal{N}(A - \lambda_i I_n)\) consists of all the eigenvectors of \(A\) with eigenvalue \(\lambda_i\) together with the zero vector, we conclude that for each \(i = 1, \ldots, k\), either \(\vec{w}^{(i)}\) is an eigenvector of \(A\) with eigenvalue \(\lambda_i\), or else \(\vec{w}^{(i)} = \vec{0}_n\).

\(^{19}\)For instance, \(\det \begin{pmatrix} 2 & 1 & 1 \\ 3 & 6 & 0 \\ 8 & 0 & 6 \end{pmatrix} = 6 \neq 0\).

\(^{20}\)I’ll wager you the national debt of Greece (in return for a freshly pressed jar of olive oil) that a randomly-selected \(n \times n\) matrix will have \(n\) distinct eigenvalues.

\(^{21}\)We’re using an important result called The Fundamental Theorem of Arithmetic, which states that a polynomial of degree \(n\) whose coefficients are real numbers, or more generally, complex numbers, must have \(n\) roots. These roots could be complex numbers. The proof of this theorem is beyond the scope of our investigation, though not beyond the scope of your understanding—it just requires membership in another cult or two. However, in light of your math journey so far, the fact that a polynomial of degree \(n\) has \(n\) roots should not be a surprise.

\(^{22}\)Counting multiplicity\(^\dagger\) means if \(\lambda = 2\) appears as a root three times, then we count it three times. So \((\lambda - 2)^3(\lambda - 1) = 0\) has four roots, 2, 2, 2, and 1.
If \( \bar{w}^{(i)} \) does not equal \( \bar{0}_n \) for at least one \( i \), then the linear combination in (\( \star \)) is a sum of eigenvectors with distinct eigenvalues; in particular, it is a nontrivial linear combination of eigenvectors with distinct eigenvalues. This contradicts the linear independence of such eigenvectors as guaranteed by Observation 18. Thus, we must have \( \bar{w}^{(i)} = \bar{0}_n \) for all \( i = 1, \ldots, k \).

But for each \( i \), \( \bar{v}^{(i)}_1, \ldots, \bar{v}^{(i)}_{t_i} \) form a basis for \( \mathcal{N}(A - \lambda_i I_n) \). Thus, since \( \bar{w}^{(i)} = a^{(i)}_1 \bar{v}^{(i)}_1 + \cdots + a^{(i)}_{t_i} \bar{v}^{(i)}_{t_i} = \bar{0}_n \), then \( a^{(i)}_1 = \cdots = a^{(i)}_{t_i} = 0 \). As this is true for all \( i \), we conclude that \( \bar{v}^{(1)}_1, \ldots, \bar{v}^{(1)}_{t_1}, \ldots, \bar{v}^{(k)}_1, \ldots, \bar{v}^{(k)}_{t_k} \) are \( n \) linearly independent eigenvectors for \( \mathbb{R}^n \). Therefore, by Observation 4, \( A \) must be diagonalizable.

Conversely, suppose that \( A \) is diagonalizable. Then Observation 4, \( A \) must have \( n \) linearly independent eigenvectors. Group these \( n \) linearly independent eigenvectors by eigenvalue, and suppose there are among them \( m_i \) eigenvectors with eigenvalue \( \lambda_i \). This means \( \mathcal{N}(A - \lambda_i I_n) \) must contain at least \( m_i \) linearly independent eigenvectors; in other words, \( m_i \leq t_i = \dim(A - \lambda_i I_n) \) for all \( i = 1, \ldots, k \). By Observation 17, \( t_i \leq s_i \) for all \( i = 1, \ldots, k \). Adding up the counts, we get

\[
\begin{align*}
\text{number of linearly independent eigenvectors} & \leq t_1 + \cdots + t_k \\
\text{number of eigenvalues} & \leq s_1 + \cdots + s_k = n
\end{align*}
\]

So we must have \( t_1 + \cdots + t_k = n \). The only way this can happen under the restriction \( t_i \leq s_i \) is if \( t_i = s_i \) for all \( i = 1, \ldots, k \). Therefore, the algebraic and geometric multiplicities are equal for each eigenvalue of \( A \).

**Example 22.** You find yourself in a fluorescent-lit room with a single desk in the middle. On the desk is a computer running some sort of computation program. The screen says:

```
sage: matrix([[-6,4,0,9],[-3,0,1,6],[-1,-2,1,0],[4,4,0,7]]).eigenvalues()
[[5, 1, -2, -2]]
```

You assume this means that the matrix
\[
\begin{pmatrix}
-6 & 4 & 0 & 9 \\
-3 & 0 & 1 & 6 \\
-1 & -2 & 1 & 0 \\
-4 & 4 & 0 & 7
\end{pmatrix}
\]
has eigenvalues 5, 1, -2, and -2. In particular, the eigenvalue -2 has algebraic multiplicity 2. You jiggle the mouse to play with the program, when the screen goes blank for a second before displaying the following question.

**Diagonalizable? [Y/N] Answer in 60 seconds to avert sinister emergency protocol**

You do not know what the emergency protocol is, but it sounds sinister. However, 60 seconds is not enough time to diagonalize this matrix. But...you don’t need to diagonalize it...you need only figure out if it is diagonalizable. The algebraic multiplicity is 1 for each of the eigenvalues 5 and 1; therefore, their geometric multiplicities equal 1 as well. The only thing you need to check is the geometric multiplicity of the eigenvalue -2; if it equals 2, then the matrix is diagonalizable according to Observation 21.

Fifty five seconds left. You spring into action solving (\( A + 2I_4 \mid \bar{0}_4 \)):

\[
\begin{pmatrix}
-4 & 4 & 0 & 9 & 0 \\
-3 & 2 & 1 & 6 & 0 \\
-1 & -2 & 3 & 0 & 0 \\
-4 & 4 & 0 & 9 & 0 \\
0 & 8 & -8 & 6 & 0 \\
0 & 12 & -12 & 9 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \rightarrow
\begin{pmatrix}
-1 & -2 & 3 & 0 & 0 \\
0 & 8 & -8 & 6 & 0 \\
0 & 12 & -12 & 9 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

The homogeneous sole has two free variables, which means \( \dim(\mathcal{N}(A + 2I_4)) = 2 \), and so the geometric multiplicity of the eigenvalue -2 equals 2 as well. The algebraic and geometric multiplicities of each of the eigenvalues are equal. Therefore, by Observation 21, the matrix is diagonalizable. Yes!

You press Y with four seconds to spare. The screen switches to Netscape, and you spend the next few hours browsing chocolate cake pics on Instagram.

## 5. Symmetric Matrices Are Great!

This section is still under heavy construction and will be updated periodically.

Some of the most prevalent matrices in nature are symmetric, and that’s a wonderful thing, because symmetric matrices are very nicely behaved as far as eigentings are concerned.

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23Recall that \( A \) is symmetric if \( A^T = A \)
Example 23. Our old friend $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ from Example 2 is symmetric. We found eigenvalues to be 3 and −1, with eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, respectively. There are two wonderful features of $A$’s eigenstuff we must highlight. First, the eigenvalues are real numbers. Second, the eigenvectors are perpendicular to each other; we can verify that by computing their dot product: $(1)(1) + (1)(1) = 0$.

5.1. Their eigenvalues are real! Recall that a complex number is of the form $a + bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. The standard arithmetic operations extend naturally to the complex numbers. As a result, so does linear algebra.

Addition and subtraction: $(2 - 5i) - (3 + 3i) = -1 - 8i$.
Multiplication: $(-2 + 3i)(4 - i) = (-2)(4) + (-2)(-i) + (3i)(4) - (3i)(i) = -8 + 2i + 12i - 3i^2 = -8 + 14i - 3(-1) = 11 + 14i$. Note as well the property $(a + bi)(a - bi) = a^2 + b^2$.
Division: $\frac{1}{2 + 3i} = \frac{1}{2 + 3i} \left(\frac{2 - 3i}{2 + 3i}\right) = \frac{2 - 3i}{4 + 9} = \frac{2}{13} - \frac{3}{13}i$.

Observation 24. All the eigenvalues of a symmetric matrix are real.

5.2. They’re Diagonalizable! We need to explore what diagonalizability isn’t in order to better understand what it is for symmetric matrices. Recall from Observation 21 that a matrix is diagonalizable precisely when the algebraic and geometric multiplicities are equal for all its eigenvalues. The next observation highlights a peculiar property of nondiagonalizable matrices; after that, we’ll show that symmetric matrices cannot have such a property.

Observation 25. Suppose $A$ is a nondiagonalizable $n \times n$ matrix. Then $A$ has an eigenvector $\tilde{v}_0$ with eigenvalue $\lambda_0$ such that $(A - \lambda_0 I_n)\tilde{v} = \tilde{v}_0$ is a consistent sole.

Proof. By Observation 21, $A$ must have an eigenvalue $\lambda_0$ with algebraic multiplicity strictly greater than geometric multiplicity $t$. By Observation 16, $A = QMQ^{-1}$, where $M$ is of the form $M = \begin{pmatrix} \lambda_0 I_t \\ 0 \end{pmatrix} C^t$.

and $Q = \begin{pmatrix} | & | & | \\ \tilde{v}_1 & \cdots & \tilde{v}_n \end{pmatrix}$, where $\tilde{v}_1, \cdots, \tilde{v}_t$ form a basis for $\mathcal{N}(A - \lambda_0 I_n)$. By Observation 17, $\lambda_0$ is an eigenvalue of $C$.

Let $\tilde{u} \in \mathbb{R}^{n-t}$ be an eigenvector of $C$ with eigenvalue $\lambda_0$; thus, $(C - \lambda_0 I_{n-t})\tilde{u} = 0_{n-t}$. Let $\tilde{w} \in \mathbb{R}^n$ be the vector whose first $t$ entries are 0, and whose last $n - t$ entries equal $\tilde{u}$. So $\tilde{w}$ is of the form $\tilde{w} = \begin{pmatrix} \tilde{0}_t \\ \tilde{u} \end{pmatrix}$.

Consider the product $(M - \lambda_0 I_n)\tilde{w}$:

$$M - \lambda_0 I_n = \begin{pmatrix} | & U \\ 0 & C - \lambda_0 I_{n-t} \end{pmatrix} \begin{pmatrix} \tilde{0}_t \\ \tilde{u} \end{pmatrix} = \begin{pmatrix} U\tilde{u} \\ (C - \lambda_0 I_{n-t})\tilde{u} \end{pmatrix} = \begin{pmatrix} U\tilde{u} \\ 0_{n-t} \end{pmatrix}$$

If $U\tilde{u} = \tilde{0}_t$, then $(M - \lambda_0 I_n)\tilde{w} = \tilde{0}_n$. In other words, $\tilde{w}$ would be an eigenvector of $M$ with eigenvalue $\lambda_0$. In that case, by Observation 13, $Q\tilde{w}$ would be an eigenvector of $A$ with eigenvalue $\lambda_0$. Because the first $t$ entries of $\tilde{w}$ equal 0, $Q\tilde{w}$ is a linear combination of $\tilde{v}_t, \cdots, \tilde{v}_n$. Since $\tilde{v}_1, \cdots, \tilde{v}_n$ are linearly independent, no linear combination of the last $n - t$ entries can be in the span of the first $t$; in particular, $Q\tilde{w} \notin \mathcal{N}(A - \lambda_0 I_n)$. Thus, $Q\tilde{w}$ is an eigenvector of $A$ with eigenvalue $\lambda_0$ but is not contained in $\mathcal{N}(A - \lambda_0 I_n)$; this contradicts Observation 3. Therefore, our assumption that $U\tilde{u} = \tilde{0}_t$ must have been false, and $(M - \lambda_0 I_n)\tilde{w}$ must be a nontrivial linear combination of $\tilde{e}_1, \cdots, \tilde{e}_t$. Say $(M - \lambda_0 I_n)\tilde{w} = a_1\tilde{e}_1 + \cdots + a_t\tilde{e}_t$, where not all the $a_i$ equal 0.

24 The symmetry of $A$ follows from the fact that the top right and bottom left entries are equal, not because the diagonal entries are equal. The matrix $\begin{pmatrix} 1 & 2 \\ 2 & 47 \end{pmatrix}$ is also symmetric.

25 Contrast that with Example 9. While nonsymmetric matrices can have real eigenvalues, symmetric matrices must.
Let \( \tilde{b} = Q\tilde{w} \), and let \( \tilde{v}_0 = (A - \lambda_0 I_n)\tilde{b} \). Then
\[
\tilde{v}_0 = (A - \lambda_0 I_n)\tilde{b} = (QM^{-1} - \lambda_0 I_n)(Q\tilde{w}) = QM\tilde{w} - \lambda_0 Q\tilde{w} = Q(M - \lambda_0 I_n)\tilde{w} = Q(a_1\tilde{v}_1 + \ldots + a_t\tilde{v}_t) = a_1\tilde{v}_1 + \ldots + a_t\tilde{v}_t
\]
Since \( \tilde{v}_1, \ldots, \tilde{v}_t \) are linearly independent, \( \tilde{v}_0 \neq 0_\mathbb{R} \). Furthermore, \( \tilde{v}_0 \in \text{span}\{\tilde{v}_1, \ldots, \tilde{v}_t\} = N(A - \lambda_0 I_n) \). That is, \( \tilde{v}_0 \) is an eigenvector of \( A \) with eigenvalue \( \lambda_0 \). Finally, \( \tilde{b} \) is a solution to the sole \( (A - \lambda_0 I_n)\tilde{x} = \tilde{v}_0 \).

Now we can show that symmetric matrices do not have this property.

**Observation 26.** Symmetric matrices are diagonalizable.

**Proof.** Suppose \( A \) is a symmetric matrix; that is, \( A^T = A \). Assume \( A \) is not diagonalizable. Then by Observation 25, \( A \) would have an eigenvector \( \tilde{v}_0 \) with eigenvalue \( \lambda_0 \) such that the sole \( (A - \lambda_0 I_n)\tilde{x} = \tilde{v}_0 \) is consistent. Let \( \tilde{b} \) be a solution to this sole. Then \( (A - \lambda_0 I_n)^2\tilde{b} = (A - \lambda_0 I_n)((A - \lambda_0 I_n)\tilde{b}) = (A - \lambda_0 I_n)\tilde{v}_0 = 0_\mathbb{R} \).

On the other hand, since \( A \) is symmetric, so is \( (A - \lambda_0 I_n) \); that is, \( (A - \lambda_0 I_n)^T = A - \lambda_0 I_n \). Since \( \tilde{v}_0 \) is a nonzero vector, it must have nonzero length. Since \( \tilde{v}_0 \) is real by Observation 24, then
\[
0 = ||\tilde{v}_0||^2 = \tilde{v}_0^T\tilde{v}_0 = ((A - \lambda_0 I_n)\tilde{b})^T((A - \lambda_0 I_n)\tilde{b}) = \tilde{b}^T(A - \lambda_0 I_n)^2\tilde{b} = \tilde{b}^T(A - \lambda_0 I_n)(A - \lambda_0 I_n)\tilde{b} = \tilde{b}^T(A - \lambda_0 I_n)\tilde{v}_0 = 0
\]
This contradiction implies our assumption must have been false. Therefore, \( A \) is diagonalizable.

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**5.3. Their eigenvectors are orthogonal! [Coming soon]**

**Exercises**

1. (a) Find the eigenvalues and eigenvectors of the matrix \( A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \).
   (b) Calculate \( A^k \).

2. Suppose \( a, b \) and \( c \) are any numbers such that \( a \neq c \) and \( b \neq 0 \). Find the eigenvectors and eigenvalues of the matrix \( T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \).

3. Show that an \( n \times n \) matrix is invertible if and only if 0 is not an eigenvalue.

4. Suppose the \( n \times n \) matrix \( P \) is a projection matrix onto some subspace \( V \) of \( \mathbb{R}^n \). Find the eigenvectors and eigenvalues of \( P \).

5. **Exploration.** Suppose \( A \) and \( B \) are two \( n \times n \) matrices. If \( \tilde{v} \) is an eigenvector of \( B \) with eigenvalue \( \lambda \), \( \tilde{v} \) is not necessarily an eigenvector of \( AB \). Give an example to illustrate that. Now, what additional hypotheses are needed for \( \tilde{v} \) to be an eigenvector of \( AB \)? In that case, what is its eigenvalue? Given the eigenvectors and eigenvalues of \( A \), describe the eigenvectors and eigenvalues of \( A^k \).

6. Diagonalize the matrix \( \begin{pmatrix} 1 & 0 & 0 \\ -8 & 4 & -6 \\ 8 & 1 & 9 \end{pmatrix} \).

7. Suppose \( A \) is a \( 2 \times 2 \) matrix with eigenvalues \( \lambda_1 = -1 \) and \( \lambda_2 = 2 \) and corresponding eigenvectors \( \tilde{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( \tilde{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \). Let \( x_0 \) and \( y_0 \). Let \( \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \).
   (a) Let \( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = A \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \). Compute \( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \) without calculating \( A \).
(b) Define \( \begin{pmatrix} x_k \\ y_k \end{pmatrix} = A \begin{pmatrix} x_{k-1} \\ y_{k-1} \end{pmatrix} \); that is, each vector in the sequence \( \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \), etc, is obtained from the one before it by multiplication by \( A \). Find a formula for \( \begin{pmatrix} x_k \\ y_k \end{pmatrix} \) in terms of \( k \), again without calculating \( A \).
(c) Calculate \( A \).

(8) (a) Verify that 3 is an eigenvalue of the matrix \( \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & -1 \end{pmatrix} \).
(b) Let \( A \) be an \( n \times n \) matrix and \( k \) a fixed number. Suppose that the sum of the entries in each row of \( A \) equals \( k \); in other words, the sum of the columns of \( A \) is the column vector whose entries are all \( k \). Show that \( k \) is an eigenvalue of \( A \).

(9) Suppose \( A \) is an invertible \( n \times n \) matrix, and \( \vec{v} \) is an eigenvector of \( A \) with eigenvalue \( \lambda \). We know from exercise 3 above that \( \lambda \neq 0 \). Show that \( \vec{v} \) is an eigenvector of \( A^{-1} \) with eigenvalue \( 1/\lambda \).

(10) Suppose \( \vec{v} \) is an eigenvector of the \( n \times n \) matrix \( A \), with associated eigenvalue \( \lambda \). Let \( c \) be any scalar. Show that \( \vec{v} \) is an eigenvector of \( A + cI_n \) with eigenvalue \( \lambda + c \).

(11) Suppose \( A \) and \( B \) are two \( n \times n \) diagonalizable matrices. Show that, if \( A \) and \( B \) have the same eigenvectors (but not necessarily the same eigenvalues), then \( AB = BA \).

(12) Prove that a \( 3 \times 3 \) matrix always has at least one real eigenvector.

(13) Show that the matrix \( \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \) is not diagonalizable.

(14) Suppose \( A = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix} \) is the matrix representation of the linear function \( L : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) with respect to the standard basis.
(a) Find an \( \mathbb{R}^2 \) basis \( B \) such that the matrix representation of \( L \) in the coordinates of \( B \) is a diagonal matrix.
(b) Plot the eigenvectors, and use the geometrical interpretation of eigenvectors (scaling, etc.) to describe the linear function \( L \).

(15) (a) Prove that the product of the eigenvalues of a matrix equals the determinant of that matrix.
You may assume the matrix is diagonalizable.
(b) Prove part (a) for non-diagonalizable matrices.

(16) Consider the \( n \times n \) matrix \( A = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix} \).
(a) Show that \( \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \) is an eigenvector of \( A \), and find the corresponding eigenvalue.
(b) Show that \( \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix} \) is an eigenvector of \( A \), and find the corresponding eigenvalue.
(c) Find eigenvectors \( \vec{v}_3, \ldots, \vec{v}_n \) in a manner analogous to part (b).
(d) Show that the eigenvectors are linearly independent; calculate \( \det(A) \).

(17) Diagonalize the \( n \times n \) matrix each of whose entries equal 1. Refer to exercise 11 of Chapter 3.
(18) Use diagonalization to find the limit of \( \left( \begin{array}{cc} 3/5 & 4/5 \\ 2/5 & 1/5 \end{array} \right)^n \) as \( n \to \infty \).

(19) Consider the sequence of vectors \( \vec{w}_0, \vec{w}_1, \ldots \) given by
\[
\vec{w}_{n+1} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \vec{w}_n.
\]
(a) Suppose \( \vec{w}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \). Calculate \( \vec{w}_n \) in terms of \( n \) and \( x_0, y_0, z_0 \).
(b) Calculate \( \vec{w}_n \) as \( n \to \infty \).

(20) Redo the previous exercise with the matrix \( \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \). This is a difficult problem. Think of sneaky approaches.

(21) Consider the sequence \( 1, 1, 2, 3, 5, 8, 13, 21, \ldots \), and denote the first term by \( f_0 \) (so that, for instance, \( f_5 = 8 \)). Let \( \vec{v}_n = \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} \).
(a) Find a \( 2 \times 2 \) matrix \( A \) such that \( A\vec{v}_{n-1} = \vec{v}_n \).
(b) Diagonalize \( A \).
(c) Find a formula for \( f_n \), and use that to compute \( \lim_{n \to \infty} \frac{f_n}{f_{n-1}} \). Compare your answer with that of Chapter 4, exercise 15.

(22) Recall exercises 20 and 21 of Chapter 4. Verify that the given basis is appropriate by finding the eigenvectors and eigenvalues of the matrix \( \begin{pmatrix} 1.5 & 0 \\ -12 & 2.6 \end{pmatrix} \).

(23) Redo exercise 14 of Chapter 1 using eigentheory.

(24) An \( n \times n \) matrix \( A \) is called symmetric if \( A^T = A \). That is, the rows of \( A \) and the columns of \( A \) are equal. For instance, the matrix \( \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \) is symmetric, as are all diagonal matrices.
(a) Show that a \( 2 \times 2 \) symmetric matrix will have real eigenvalues.
(b) Show that a \( 2 \times 2 \) symmetric matrix will always be diagonalizable. You may want to divide this problem into two cases, depending on whether the eigenvalues are distinct or not.

(25) (a) Find all solutions to the linear system of differential equations
\[
\begin{align*}
f'(x) &= f(x) + g(x) \\
g'(x) &= -2f(x) - 4g(x)
\end{align*}
\]
(b) Find the specific solution for which \( f(0) = 2 \) and \( g(0) = 1 \).

(26) (a) Suppose \( A \) is an \( n \times n \) matrix for which \( A^2 = A \); for instance, projection matrices satisfy that property. Show that the eigenvalues of \( A \) can only equal 0 or 1.
(b) Suppose \( B \) is an \( n \times n \) matrix for which \( B^2 = I_n \); for instance, reflection matrices satisfy that property. Show that the eigenvalues of \( B \) can only equal 1 or \(-1\).

(27) Find the eigenvalues and eigenvectors of the matrix \( \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \).

(28) Use the power method to approximate the dominant eigenvalue and eigenvector of the matrix
\[
\begin{pmatrix} 9 & 4 & 8 \\ 4 & 15 & -4 \\ 8 & -4 & 9 \end{pmatrix}.
\]
(29) Consider the $n \times n$ matrices $Q$ (invertible) and $D$ (diagonal), and let $A = QDQ^{-1}$. Let $\tilde{v}_k$ be the $k^{\text{th}}$ column of $Q$. Show that $\tilde{v}_k$ is an eigenvector of $A$ whose associated eigenvalue is the $k^{\text{th}}$ diagonal entry in $D$. This result says that if $A$ looks like it’s diagonalized, then it is.

(30) **Prove; or, disprove, and salvage if possible.**

(a) Suppose $A$ and $B$ are two $n \times n$ matrices. If $\tilde{v}$ is an eigenvector of $A$, then $\tilde{v}$ is an eigenvector of $BA$.

(b) Suppose $A$ is an $n \times n$ matrix. If $\lambda$ is an eigenvalue of $A$, then $\lambda$ is an eigenvalue of $A^T$.

(c) Suppose $\tilde{v}$ is an eigenvector of the matrix $A$, with eigenvalue $\lambda$. Let $p(x)$ be a polynomial. Then $\tilde{v}$ is an eigenvector of $p(A)$ with eigenvalue $p(\lambda)$.

(d) A $2 \times 2$ matrix cannot have one real eigenvalue and one complex eigenvalue.

(31) **Exploration.** Since $\begin{pmatrix} 1 & \sqrt{2} \\ 3 & 4 \end{pmatrix}^2 = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$, we can say that the square root of $\begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$ is $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. When do $2 \times 2$ matrices have square roots?

(32) A square matrix $A$ is called **skew-symmetric** if $A^T = -A$. For instance, the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is skew-symmetric. Show that if $n$ is odd, then the determinant of an $n \times n$ skew-symmetric matrix equals zero. Conclude that $\lambda = 0$ is always an eigenvalue of such a matrix.

(33) **Exploration.** A square matrix $A$ is called **orthogonal** if $A^T = A^{-1}$. For instance, $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ is orthogonal. What’s so special about the eigenvectors and eigenvalues of orthogonal matrices?