CS131 Lambda Calculus Worksheet

Due at the beginning of class on Tuesday, April 3rd

Name: _____

CAS ID (e.g., abc01234@pomona.edu): _____

I encourage you to collaborate. Please record your collaborations below.

Except for the problems in Section 9, you should use equational reasoning rather than CBN or CBV reduction.

Unless the problem says otherwise, feel free to use any definition from class or in the lecture notes, like `true` and `succ`.

Each problem is worth one point, except problems marked with a (C) are "challenge" problems worth no points—go ahead and test your mettle, but these are longer or harder than anything I'd put on an exam.

Please turn in your work as a printout of this sheet, not on separate paper. If you would rather typeset your work, I can give you the LaTeX... but you'll learn more by writing it by hand.

Collaborators: _____

1 Understanding syntax

1.1 I think that I shall never // a syntax lovely as a tree

Consider the lambda term $\lambda x.\ y\ (\lambda z.\ z)\ x$. Draw it as a tree where the leaves are variables and the branches are lambdas and applications.

1.2 Tilting at windmills

Consider the lambda term $\lambda f.\ \lambda x.\ f\ (x\ x)$. Draw it as a tree where the leaves are variables and the branches are lambdas and applications.

2 Binding variables

2.1 I'm not a number, I'm a free variable!

Circle the free variables in the following lambda calculus terms.

$x\ x$	$x\ y$	$\lambda x.\ x$
$\lambda x.\ y$	$\lambda x.\ x\ x$	$\lambda x.\ x\ x\ x$

2.2 An open and shut case

For each of the following lambda calculus terms, identify whether it is open (i.e., has free variables) or closed—circle the appropriate answer.

1. $a\ b\ c$ open closed
2. $a\ (b\ c)$ open closed
3. $\lambda x.\ x$ open closed
4. $\lambda x.\ y$ open closed
5. $\lambda a.\ \lambda b.\ a$ open closed
6. $\lambda q.\ r\ (\lambda r.\ q)$ open closed
7. $\lambda xyz.\ z$ open closed
8. $\lambda x.\ \lambda y.\ \lambda z.\ z$ open closed
9. $\lambda n.\ \lambda sz.\ s\ (n\ s\ z)$ open closed

3 Substitution

3.1 Changing names

Substitute y for x in $\lambda z.\ x$, i.e., compute $(\lambda z.\ x)[y/x]$.

3.2 Accept no substitutes

Substitute y for x in $\lambda z.\ z$, i.e., compute $(\lambda z.\ z)[y/x]$.

3.3 Identity crisis

Substitute $\lambda y.\ y$ for x in $\lambda x.\ x$, i.e., compute $(\lambda x.\ x)[\lambda y.\ y/x]$.

3.4 Don't get carried away

Substiute $\lambda x.\ x\ x$ for x in $\lambda x.\ x\ x$, i.e., compute $(\lambda x.\ x\ x)[\lambda x.\ x\ x/x]$.

4 Alpha renaming and beta reduction

4.1 Explicit content

Use $=_\beta$ to reduce $(\lambda x\ y.\ x)\ (\lambda y.\ y)\ (\lambda x.\ x)$ as much as possible. Use $=_\alpha$ to avoid potential name conflicts.

Perform the same reduction without using $=_\alpha$.

4.2 I come from a land down under

Reduce $\lambda x.\ (\lambda y.\ y)\ x$ as much as possible.

4.3 You can count on me

Reduce $(\lambda s\ z.\ s\ (s\ (s\ z)))\ (\lambda x.\ i\ x)\ n$ as much as possible; since i and n are free variables, you may not end up with a closed term.

5 Booleans

5.1 Ask not what not can do for you, but what you can do for not

Write not (a/k/a negation, ¬, !) on Church booleans without using anything other than lambdas, variables, and applications.

Write not a different way, using existing definitions.

5.2 Either/or

Write xor (a/k/a exclusive-or, ⊗, ^) on Church booleans without using anything other than lambdas, variables, and applications.

Write xor a different way, using existing definitions.

5.3 Any boolean you like

Write a function corresponding to the predicate $P(x, y, z) = (x \wedge y) \vee (\neg x \wedge z) \vee (\neg x \wedge \neg y \wedge \neg z)$.

5.4 Choices, choices

Write a function of two arguments. Its second argument will always be a Church boolean. Your function should (a) return its first argument if its second argument is true and (b) return $\lambda x.\ x$ if its second argument is false.

6 Church numerals

6.1 Don't be a □

Write a function that takes in a Church numeral n and squares it, returning n^2.

6.2 Polynomial want a cracker?

Write a function that takes in Church numerals x, y, and z and returns the value of $y^3 + 3x^2z^2 + 2z + 5$.

6.3 With great power comes great responsibility

Write exponentiation, i.e., a function that takes as arguments two Church numerals m and n and returns m^n.

6.4 Toe the line

Write a function that takes in a slope m, a y-intercept b, and a Church pair of numbers x and y; return true if $y = mx + b$ and false otherwise.

6.5 Binomial, save later (C)

Reminder: problems marked with (C) are 'challenge' problems that won't be graded.

Write a function binomial that takes Church numerals x, y, and n and computes $(x + y)^n$ according to the Binomial Theorem:

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

You can assume you have choose from Problem 7.2. Don't use the Y combinator.

7 Recursion

7.1 Why ask Y?

Write the Fibonacci function.

7.2 I choose you, Pikachu!

Write a function choose that takes Church numerals m and n and computes $\binom{m}{n}$. You'll need division, but you can do this problem without having solved Problem 7.3—just assume you have divide.

It was super effective!

7.3 Divide and conquer (C)

Write divide on Church numerals.

Write a function divides that checks divisibility, i.e., it takes two Church numerals m and n and returns true if m divides n and false otherwise.

7.4 Go ahead and be negative (C)

Church numerals represent the natural numbers ($\mathbb{N} = \{0, 1, \ldots\}$). Use Church numerals, ingenuity, and elbow grease to represent the integers ($\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$). Define zzero, zsucc, ziszero, zpred, zplus, zminus, ztimes, and zequal.

8 Lists

Let's define lists as follows:

$$\begin{array}{rcl}
\mathsf{nil} & = & \lambda n\ c.\ n \\
\mathsf{cons} & = & \lambda h\ t.\ \lambda n\ c.\ c\ h\ t \\
\mathsf{null} & = & \lambda l.\ l\ \mathsf{true}\ (\lambda h\ t.\ \mathsf{false}) \\
\mathsf{head} & = & \lambda l.\ l\ \Omega\ (\lambda h\ t.\ h) \\
\mathsf{tail} & = & \lambda l.\ l\ \Omega\ (\lambda h\ t.\ t) \\
\mathsf{foldr} & = & \mathsf{Y}\ (\lambda \mathsf{foldrRec}.\ \lambda f\ b\ l.\ l\ b\ (\lambda h\ t.\ f\ h\ (\mathsf{foldrRec}\ f\ b\ t)))
\end{array}$$

8.1 A one, a two, a one two three four

Write down the list $[1, 2, 3, 4]$, i.e., the list containing the Church numerals one, two, three, and four in that order. Use nil and cons.

Write down the same list "directly": don't use anything but lambdas, variables, applications, and the Church numerals.

8.2 Two heads are better than one

Write down a function that returns the second element of a list. Your function should diverge if the list is too short.

8.3 One-by-one

Write map without using foldr.

Write map using foldr.

8.4 To the left, to the left

Write foldl.

8.5 I prefer French press

Write filter without using foldr.

Write filter using foldr.

8.6 Line 'em up

Write a function that takes a number n and produces the list $[1, 2, 3, \ldots, n]$. Return the empty list if n is zero.

8.7 Panning for gold (C)

Write a function that takes a number n and returns a list of all primes less than or equal to n. You'll need the divides predicate from Problem 7.3 in addition to the list functions we've just defined. Good luck!

9 You say CBN, I say CBV—let's call the whole thing off

9.1 Stop right there

How many steps does it take $(\lambda x\ y.\ y)$ (succ zero) (succ one) to reduce to a value using call-by-name? Show your work.

How many steps does it take $(\lambda x\ y.\ y)$ (succ zero) (succ one) to reduce to a value using call-by-value? Show your work.

9.2 One of these things is not like the other

Write a term that diverges in CBV but not in CBN.

9.3 Common ground

Write a term that diverges in both CBV and CBN. Don't just write Ω: the term should take at least three steps in either evaluation scheme before looping.

9.4 CBV Church booleans (C)

In CBV evaluation, we evaluate all of a function's arguments before β-reducing. Define a version of the Church booleans (true, false, and, or, and not) and some syntactic sugar for if expressions that work in CBV evaluation. For example, the syntactic sugar for let expressions is let $x = e_1$ in $e_2 \equiv (\lambda x.\ e_2)\ e_1$.

9.5 Loop the loop (C)

In CBV, $\Omega = (\lambda x.\ x\ x)\ (\lambda x.\ x\ x)$ reduces to itself in *one* step. Write a term that reduces to itself in *two* steps.

Write a term that reduces to itself in *three* steps.

Given a number $n > 1$, what would a term that reduces to itself in n steps look like?

Does your term also work for CBN? If not, write one that does.