CS 055 Spring 2017 Sample Midterm

Wednesday, March 8th

Question	Score	Points
Propositions		7
Truth tables		5
Proofs		3
Induction		6
Total		21

THIS IS THE KEY

This is the sample midterm, and is not for a grade.

You may use the proof handout I distributed, but nothing else.

Please do not cite previously proven lemmas of any origin (class, book, Internet).

Question 1: Propositions *(7 points)*

Convert each of the following to a logical proposition. To get full credit, you must convert the proposition *all the way*, i.e., there should be no English words left in your proposition.

(a) (2 points) The relation $R \subseteq B \times B$ is transitive.

$\forall xyz,\ x\ R\ y \wedge y\ R\ z \Rightarrow x\ R\ z$

(b) (2 points) The set S is a proper subset of T, i.e., it's contained in but not equal to T.

$S \subsetneq T$

(c) (3 points) There is only one number less than 1 in the naturals.

$\exists! n \in \mathbb{N}, n < 1$

Question 2: Truth tables *(5 points)*

Use a truth table to prove that $\neg(\neg p)$ is equivalent to p.

Solution:

p	$\neg p$	$\neg(\neg p)$
$\top$	$\bot$	$\top$
$\bot$	$\top$	$\bot$

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1 pt for drawing a truth table
1 pt for p
1 pt for neg p
1 pt for neg neg p
1 pt for right answer
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Question 3: Proofs *(3 points)*

For these multiple-choice questions about proof, you have to identify what (if anything) is wrong with the given proofs. Be careful: some of these proofs may be wrong *even though the theorem is true.*

Please fill in the circle next your answer. Any other marks will be ignored.

Do not guess. Leaving a question unanswered will be worth half a point, while a wrong answer is worth no points.

(a) (1 point) **Theorem:** $A \cup B = B$ iff $A \subseteq B$.

Proof: Let sets A and B be given. Suppose $A \cup B = B$; we must show that $A \subseteq B$. To have $A \cup B = B$, that means that $x \in A \cup B$ iff $x \in B$. We want to show that $A \subseteq B$, i.e., if $x \in A$ then $x \in B$. Suppose we have $x \in A$. We therefore have $x \in A \cup B$; by assumption, we therefore have $x \in B$. $\square$

○ This proof is correct.

○ This proof is wrong: this property should be proved element-wise.

○ This proof is wrong: we may not have $x \in A$, since A could be empty.

√ **This proof is wrong: it only proves one direction.**

○ This proof is wrong: $\cup$ may not be well defined on A and B.

(b) (1 point) **Theorem:** $h(n) \leq n$ where we define $h : \mathbb{N} \to \mathbb{N}$ as follows:

$$h(n) = \begin{cases} 1 + h(n-2) & n \geq 2 \\ 0 & n < 2 \end{cases}$$

Proof: We go by strong induction on n. Our IH is that $\forall k < n, h(k) \leq k$. By cases on n.

($n = 0$) We have $h(0) = 0 \leq 0$.

($n = n' + 1$) By cases on n'.

($n' = 0$) We have $n = n' + 1 = 0 + 1 = 1$, so $h(n) = 0 \leq 1$.

($n' = n'' + 1$) We have $n = (n'' + 1) + 1$, so $n \geq 2$. We therefore have $h(n) = 1 + h(n-2)$; by the IH on $n - 2$, we know $h(n-2) \leq n - 2$, so $1 + n - 2 \leq n$. $\square$

√ **This proof is correct.**

○ This proof is wrong: you don't need strong induction.

○ This proof is wrong: $h(n)$ isn't well defined.

○ This proof is wrong: the IH can't be used that way.

○ This proof is wrong: you can't have nested cases in a proof.

(c) (1 point) **Theorem:** The set $\mathbb{Z}$ is countable.

Proof: We define $f : \mathbb{Z} \to \mathbb{N}$ as follows:

$$f(z) = \begin{cases} 2z & z > 0 \\ -2z + 1 & z < 0 \end{cases}$$

We must show that f is injective. Let z_1 and z_2 be given such that $f(z_1) = f(z_2)$; we must show that $z_1 = z_2$. We consider how f treats the z_i:

($z_1 > 0$, $z_2 > 0$) We have $f(z_1) = 2z_1$ and $f(z_2) = 2z_2$; we must have $z_1 = z_2$.

($z_1 > 0$, $z_2 < 0$) We have $f(z_1) = 2z_1$ and $f(z_2) = -2z_2 + 1$. Note that the former is even and the latter is odd, so it's a contradiction to have $f(z_1) = f(z_2)$.

($z_1 < 0$, $z_2 > 0$) We have $f(z_1) = -2z_1 + 1$ and $f(z_2) = 2z_2$. Here the former is odd and the latter is even, so it's a contradiction to have $f(z_1) = f(z_2)$.

($z_1 < 0$, $z_2 < 0$) We have $f(z_1) = -2z_1 + 1$ and $f(z_2) = -2z_2 + 1$; if $-2z_1 + 1 = -2z_2 + 1$, then it must be that $z_1 = z_2$. $\square$

- ○ This proof is correct.
- ○ This proof is wrong: you need to use a surjective function to prove countability.
- ○ This proof is wrong: the middle two cases aren't actually contradictory.
- ○ This proof is wrong: there are more integers than there are natural numbers.
- √ **This proof is wrong: the function f isn't total.**

Question 4: Induction *(6 points)*

Prove that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

Solution: By induction on n.

$(n = 0)$ We have $\sum_{i=0}^{0} i = 0 = \frac{0 \cdot 1}{2}$.

$(n = n' + 1)$ Our IH is $\sum_{i=0}^{n'} i = \frac{n'(n'+1)}{2}$. We compute:

$$\begin{aligned}
& \sum_{i=0}^{n'+1} i \\
= \quad & n' + 1 + \sum_{i=0}^{n'} i \\
= \quad & n' + 1 + \tfrac{n'(n'+1)}{2} && \text{(IH)} \\
= \quad & n' + 1 + \tfrac{n'^2+n'}{2} \\
= \quad & \tfrac{2n'+2}{2} + \tfrac{n'^2+n'}{2} \\
= \quad & \tfrac{n'^2+3n'+2}{2} \\
= \quad & \tfrac{(n'+1)(n'+2)}{2} \\
= \quad & \tfrac{n(n+1)}{2}
\end{aligned}$$

$\square$

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1 pt for induction
1 pt for having both cases
1 pt for correct base case
1 pt for stating IH
1 pt for correct inductive case
1 pt for overall correctness
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