CS054: Truth tables

The goal of this worksheet is to give you practice with truth tables: what are truth tables and how do you construct them? It's not for a grade—no need to turn it in! I'll post solutions, but you'll get the most out of it if you don't peek.

For all of these questions, I'll use programmatic notation—`&&` for `andb` in Coq, `||` for `orb` in Coq, and `!` for `negb`. We write t for `true` and f for `false`; you can also use T and F or ⊤ and ⊥. The questions are asking about the definitions we have in `Basics.v`.

I expect you to simply *know* the truth tables for `||`, `&&`, `!`, and ⇒.

1. **Sample:** Consider the boolean expression `!a || b` given booleans a and b.

   (a) How many rows will the truth table have? **Answer:** four, because there are two variables, a and b, and we must consider each value they consider.

   (b) How many columns will the truth table have and what are they? **Answer:** four; one each for a and b, one for `!a`, and one for the whole expression.

   (c) What is the truth table? **Answer:**

a	b	`!a`	`!a || b`
t	t	f	t
t	f	f	f
f	t	t	t
f	f	t	t

   (d) Can you rephrase this expression in terms of other boolean operators? That is, can you find a smaller boolean expression that has an equivalent truth table? **Answer:** The expression $a \Rightarrow b$ (i.e., `impb a b`) has the same truth table.

2. Consider the boolean expression `p || (p && q)`.

   (a) How many rows will the truth table have?

   > **Solution:** Four, because there are two variables.

   (b) How many columns will the truth table have?

   > **Solution:** Four: one for each variable (2), one for `p && q`, and one for the outer expression.

   (c) What is the truth table?

   > **Solution:**
   >
   > | p | q | `p && q` | `p || (p && q)` |
   > |---|---|---|---|
   > | t | t | t | t |
   > | t | f | f | t |
   > | f | t | f | f |
   > | f | f | f | f |

   (d) Can you rephrase this expression in terms of other boolean operators? That is, can you find a smaller boolean expression that has an equivalent truth table?

> **Solution:** The formulae $p \; || \; (p \; \&\& \; q)$ has the same truth table as the formula p itself.

3. Consider the boolean expression $x \; \&\& \; (x \; || \; y)$.

   (a) How many rows will the truth table have?

   > **Solution:** Four, because there are two variables.

   (b) How many columns will the truth table have?

   > **Solution:** Four: one for each variable (2), one for $x \; || \; y$, and one for the outer expression.

   (c) What is the truth table?

   > **Solution:**
   >
   > | x | y | $x \; || \; y$ | $x \; \&\& \; (x \; || \; y)$ |
   > |-----|-----|---------------|------------------------------|
   > | t | t | t | t |
   > | t | f | t | t |
   > | f | t | t | f |
   > | f | f | f | f |

   (d) Can you rephrase this expression in terms of other boolean operators? That is, can you find a smaller boolean expression that has an equivalent truth table?

   > **Solution:** The formula $x \; \&\& \; (x \; || \; y)$ has the same truth table as the formula x itself.

4. Consider the boolean expression $(x \; \&\& \; y) \; || \; (x \; \&\& \; z)$.

   (a) How many rows will the truth table have?

   > **Solution:** Eight, because there are three variables.

   (b) How many columns will the truth table have?

   > **Solution:** Six: one for each variable (3), one for $x \; \&\& \; y$, one for $x \; \&\& \; z$, and one for the outer expression.

   (c) What is the truth table?

   > **Solution:**
   >
   > | x | y | z | $x \; \&\& \; y$ | $x \; \&\& \; z$ | $(x \; \&\& \; y) \; || \; (x \; \&\& \; z)$ |
   > |-----|-----|-----|-----------------|-----------------|---|
   > | t | t | t | t | t | t |
   > | t | t | f | t | f | t |
   > | t | f | t | f | t | t |
   > | t | f | f | f | f | f |
   > | f | t | t | f | f | f |
   > | f | t | f | f | f | f |
   > | f | f | t | f | f | f |
   > | f | f | f | f | f | f |

(d) Can you rephrase this expression in terms of other boolean operators? That is, can you find a smaller boolean expression that has an equivalent truth table?

> **Solution:** The formula $(x\,\&\&\,y)\,||\,(x\,\&\&\,z)$ has the same truth table as the formula $x\,\&\&\,(y\,||\,z)$. The latter formula has one fewer column in its truth table and one fewer function call, so it's "smaller".

5. Consider the boolean expression $!x\,\&\&\,!y$.

(a) How many rows will the truth table have?

> **Solution:** Four, because there are two variables.

(b) How many columns will the truth table have?

> **Solution:** Five: one for each variable (2), one for the negation of each variable, and one for the outer expression.

(c) What is the truth table?

> **Solution:**
>
> | x | y | $!x$ | $!y$ | $!x\,\&\&\,!y$ |
> |---|---|---|---|---|
> | t | t | f | f | f |
> | t | f | f | t | f |
> | f | t | t | f | f |
> | f | f | t | t | t |

(d) Can you rephrase this expression in terms of other boolean operators? That is, can you find a smaller boolean expression that has an equivalent truth table?

> **Solution:** The formula $!x\,\&\&\,!y$ has the same truth table as the formula $!(x\,||\,y)$, which has one fewer column and one fewer function call, so it's "smaller".

6. Consider the boolean expression $!a\,||\,!b$.

(a) How many rows will the truth table have?

> **Solution:** Four, because there are two variables.

(b) How many columns will the truth table have?

> **Solution:** Five: one for each variable (2), one for the negation of each variable, and one for the outer expression.

(c) What is the truth table?

> **Solution:**
>
> | a | b | $!a$ | $!b$ | $!a\,||\,!b$ |
> |---|---|---|---|---|
> | t | t | f | f | f |
> | t | f | f | t | t |
> | f | t | t | f | t |
> | f | f | t | t | t |

(d) Can you rephrase this expression in terms of other boolean operators? That is, can you find a smaller boolean expression that has an equivalent truth table?

> **Solution:** The formula `!a && !b` has the same truth table as the formula `!(a && b)`, which has one fewer column and one fewer function call, so it's "smaller".

7. Use a truth table to prove that `||` is commutative, i.e., $p \mid\mid q$ is the same as $q \mid\mid p$.

Solution:

p	q	$p \mid\mid q$	$q \mid\mid p$
t	t	t	t
t	f	t	t
f	t	t	t
f	f	f	f

8. Use a truth table to prove that `||` is associative, i.e., $p \mid\mid (q \mid\mid r)$ is the same as $(p \mid\mid q) \mid\mid r$.

Solution:

p	q	r	$p \mid\mid q$	$q \mid\mid r$	$p \mid\mid (q \mid\mid r)$	$(p \mid\mid q) \mid\mid r$
t	t	t	t	t	t	t
t	t	f	t	t	t	t
t	f	t	t	t	t	t
t	f	f	t	f	t	t
f	t	t	t	t	t	t
f	t	f	t	t	t	t
f	f	t	f	t	t	t
f	f	f	f	f	f	f

Other good practice exercises (for which no solutions will be provided):

- Use truth tables to prove that `&&` and $\otimes$ (a/k/a `xorb`) are commutative and associative.
- Use truth tables to prove that `!` is involutive, i.e., `! (! b)` is equivalent to b.
- Is $\Rightarrow$ (a/k/a `impb`) commutative or associative? Use truth tables to prove or disprove it.