Lecture 31: Inference

CS 181O
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Some slides based on those of Christina Unger

Resolution

- Resolution rule: $\dfrac{\alpha \vee \beta,\ \neg\beta \vee \gamma}{\alpha \vee \gamma}$
- Resolution refutation: (*proof by contradiction)*
  - Convert all sentences to conjunctive normal form
  - Negate the conclusion (in CNF)
  - Apply resolution rule until either
    - Derive false (contradiction!)
    - Can't apply it any more.
- Resolution refutation is sound & complete:
  - If contradiction then valid, else not valid

Predicate Logic

- Can be extended to predicate logic using unification.
  - Express all formulas in prenex form (pull all quantifiers to front) and insides in CNF.
  - Replace existential quantifiers by Skolem functions/ constants:
    - $\forall X.\ \exists Y.\ person(X) \rightarrow has(X, Y) \wedge heart(Y)$ *replaced by*
    - $person(X) \rightarrow has(X, f(X)) \wedge heart(f(X))$
    - Function f(X) has parameter X because ∃ inside ∀X
  - Use unification and resolution to get contradiction

Unification

- Unify following sentences:
  - $P(X,tony) \wedge Q(george, X, Z)$
  - $P(f(tony),tony) \wedge Q(B,C,maggie)$
- Substitution:
  - $X \mapsto f(tony),\ B \mapsto george,\ C \mapsto f(tony),\ Z \mapsto maggie$
- Use resolution, using unification to make opposites match.

Example

- John owns a dog. Every person who owns a dog, pets it. John is a person. Therefore John pets a dog.
  - ∃x.(Dog(x) ∧ Owns(j,x)),
  - ∀z∀y(Person(z) ∧ Dog(y) ∧ Owns(z,y) → Pets(z,y))
  - Person(j)
  - ¬(∃w.(Dog(w) ∧ Pets(j,w))) ≡ ∀w(¬Dog(w) ∨ ¬Pets(j,w))

Example

- John owns a dog. Every person who owns a dog, pets it. John is a person. Therefore John pets a dog.
  1. Dog(d) — *d is Skolem constant*
  2. Owns(j,d))
  3. (¬Person(x) ∨ ¬Dog(y) ∨ ¬Owns(z,y) ∨ Pets(z,y))
  4. Person(j)
  5. ¬Dog(w) ∨ ¬Pets(j,w)
- Unify and use resolution!
  - y ↦ d, w ↦ d, z ↦ j

Semantic Tableaux

Semantic Tableaux

- Search for an assignment of truth values that makes collection of formulas true.
- Proof by contradiction: Negate conclusion and show no way to make all true.
- Differ from resolution by not requiring conversion to CNF.

Rules

	conjunction			*disjunctive*	
α	α_1	α_2	β	β_1	β_2
$P \wedge Q$	P	Q	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$
$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$P \vee Q$	P	Q
$\neg(P \rightarrow Q)$	P	$\neg Q$	$P \rightarrow Q$	$\neg P$	Q

Rules

- Apply rules from table to simplify
  - If have $\neg\neg P$, replace by P
  - If have conjunctive rule on α, replace α by both α_1 and α_2
  - If have disjunctive rule on β replace by two possible branches β_1 and β_2

$$\frac{\neg\neg P}{P} \qquad \frac{\alpha}{\begin{matrix}\alpha_1 \\ \alpha_2\end{matrix}} \qquad \frac{\beta}{\beta_1 \quad \beta_2}$$

Example

- Prove $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \rightarrow R$

$$\neg((P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \rightarrow R)$$

Example

- Prove $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \rightarrow R$

$$\cancel{\neg((P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \rightarrow R)}$$

$$(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R)$$

$$\neg R$$

Example

- Prove $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \rightarrow R$

~~$\neg((P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \rightarrow R)$~~

~~$(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R)$~~

$\neg R$

$P \vee Q$

$P \rightarrow R$

$Q \rightarrow R$

Example

- Prove $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \rightarrow R$

$\neg((P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \rightarrow R)$

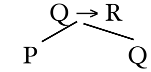
$(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R)$

$\neg R$

~~$P \vee Q$~~

$P \rightarrow R$

$Q \rightarrow R$



Example

- Prove $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \rightarrow R$

$\neg((P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \rightarrow R)$

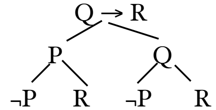
$(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R)$

$\neg R$

~~$P \vee Q$~~

~~$P \rightarrow R$~~

$Q \rightarrow R$



Example

- Prove $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \rightarrow R$

$\neg((P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \rightarrow R)$

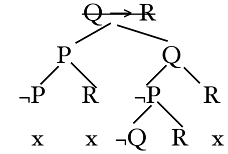
$(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R)$

$\neg R$

~~$P \vee Q$~~

~~$P \rightarrow R$~~

~~$Q \rightarrow R$~~



Example

- Prove $(P \lor Q) \land (P \to R) \land (Q \to R) \to R$

$\neg((P \lor Q) \land (P \to R) \land (Q \to R) \to R)$

$(P \lor Q) \land (P \to R) \land (Q \to R)$

$\neg R$

$P \lor Q$

$P \to R$

$Q \to R$

P / Q

$\neg P$ R | $\neg P$ R

x x | $\neg Q$ R x

x x

Inconsistent!!

Predicate Rules

universal *existential*

γ	γ_1	δ	δ_1
$\forall x.F$	$[x \to d]F$, d any name	$\exists x.F$	$[x \to d]F$, d a new name
$\neg\exists x.F$	$[x \to d](\neg F)$, d any name	$\neg\forall x.F$	$[x \to d](\neg F)$, d a new name

Rules

- Apply rules from table to simplify
  - If have $\neg\neg P$, replace by P
  - If have conjunctive rule on α, replace α by both α_1 and α_2
  - If have disjunctive rule on β replace by two possible branches β_1 and β_2
  - If have universal rule on γ, replace γ by $\gamma(d)$ for any d
  - If have existential rule on δ, replace δ by $\delta(d)$ for a new d

$\frac{\neg\neg P}{P}$ $\frac{\alpha}{\alpha_1 \; \alpha_2}$ $\frac{\beta}{\beta_1 \quad \beta_2}$ $\frac{\gamma}{\gamma(d)}$ $\frac{\delta}{\delta(d)}$ *for new d*

Example

Prove: $(\forall x)(T(x,b) \to T(a,b)), \neg T(a,b) \vdash \neg(\exists x)T(x,b)$

1. $(\forall x)(T(x,b) \to T(a,b))$ (premise)
2. $\neg T(a,b)$ (premise)
3. $\neg\neg(\exists x)T(x,b)$ (premise) *Negation of conclusion!*
4. $(\exists x)T(x,b)$ (3, $\neg\neg$)
5. $T(c,b)$ (4, Exist) *c must be new*
6. $T(c,b) \to T(a,b)$ (1, Universal)
   / \
7. $\neg T(c,b)$ $T(a,b)$ (1, Imply disjunctive)
8. x x (2,5,7, contradiction)

Soundness

- Theorem: If F is consistent, then any tableau for F will have an open branch.
- Therefore if no open branches, then must be inconsistent.
  - Recall $\Gamma \cup \{\neg\gamma\}$ inconsistent iff $\Gamma \vdash \gamma$

Questions?