Lecture 26: Gödel Incompleteness

CSCI 101
Spring, 2019

Kim Bruce

Entscheidungsproblem

- Does there exist an algorithm to decide, given a set of axioms, whether a given statement is a theorem?
  - *Church/Turing: No! FOL not decidable, but is semi-decidable*
- Is it possible to axiomatize all of the mathematical structures of interest in such a way that every true statement becomes a theorem?
  - *Allow the set of axioms to be infinite, but it must be decidable.*
  - *Gödel: No: Incompleteness theorem — fails to be semi-decidable!*

Paradoxes?

- This statement is not true.
- This statement is not provable.
  - Assume all provable statements are true
- Second is at heart of Gödel Incompleteness.

Definitions

- Let T be a decidable set of statements and let ϕ be a formula of first-order logic.
  - $T \vdash \phi$ means there is a proof of ϕ using statements of T as axioms
  - $T \vDash \phi$ means for every model in which all statements of T are true, then ϕ must be true as well.
- Example: Let T be axioms of number theory, and ϕ be $\forall x.\ \exists y.\ y > x$

Gödel Incompleteness

- T is consistent iff all provable statements are true. ($T \vdash \phi \Rightarrow T \vDash \phi$).
  - Equivalently, no false statement has a valid proof
- T is complete iff every true statement has a valid proof. ($T \vDash \phi \Rightarrow T \vdash \phi$).
- Gödel Completeness:
  In predicate logic, if $T \vDash \phi$, then $T \vdash \phi$.

Incompleteness Theorems

- Different notion of completeness -- w.r.t. model
- Gödel Incompleteness 1: For every "interesting" system there are true statements that cannot be proved.
- Gödel Incompleteness 2: For every "interesting" system, the consistency of that system cannot be proved within itself.

Interesting systems include number theory and set theory

- Axioms:
  - PA0. $\forall x \neg (0 = s(x))$
  - PA1. $\forall x \forall y (s(x) = s(y) \rightarrow x = y)$
  - PA2. $\forall x (x + 0 = x)$
  - PA3. $\forall x \forall y (x + s(y) = s(x+y))$
  - PA4. $\forall x (x \cdot 0 = 0)$
  - PA5. $\forall x \forall y (x \cdot s(y) = (x \cdot y) + x)$
  - PA6. $\phi[0/x] \wedge \forall x (\phi \rightarrow \phi[s(x)/x]) \rightarrow \forall x \phi$

Induction schema: ∞ number of rules

Completeness & Consistency

- Every provable statement of PA is true of the natural numbers.
- What about completeness?
  - Is PA enough to prove all true statements in N?
- Thm: The set of statements provable from PA is semidecidable.
- Already showed predicate logic not decidable.

Incompleteness

- Let Th(N) = set of all sentences in language of PA that are true in N
- Lemma: Th(N) is not semi-decidable. Thus $\{\phi : PA \vdash \phi\} \subset Th(N)$, but not equal.
- Proof: Show Th(N) semi-decidable implies $\neg H_{TM}$ is semi-decidable.
  - Given <M,w>, construct γ s.t.
    <M,w> $\in \neg H_{TM}$ iff $\gamma \in Th(N)$

Constructing γ

- Can encode TM computations as integers:
  - Give characters an integer code and code sequence as $2^{c1}\, 3^{c2}\, 5^{c3}\, 7^{c4}\, 11^{c5}\, 13^{c6}\, 17^{c7} \ldots$
  - Can write formula $ValidComp_{M,w}(y)$ that says y represents a valid computation history of M on input w.
  - Define $\gamma = \neg\exists y\, ValidComp_{M,x}(y)$ *says M,w not halt!*
  - As desired, get <M,w> $\in \neg H_{TM}$ iff $\gamma \in Th(N)$
  - Thus Th(N) not semidecidable, so $\{\phi : PA \vdash \phi\} \subset Th(N)$ and hence PA incomplete -- can't prove all ϕ true in N.

Gödel Incompleteness

- Gödel 1: Let T be a decidable set of axioms true of the natural numbers & that implies the axioms of Peano Arithmetic. Then there is a sentence γ which is true of N but is not provable in T.
  - Proof only depended on ability to encode computation.
  - Set of statements provable from a decidable T is semi-decidable, but Th(N) is not.
  - T consistent $\Rightarrow$ Provable(T) = $\{\phi \mid T \vdash \phi\} \subseteq Th(N)$.