Homework 7

Due midnight, Thursday, 3/26/2015

Please submit your homework solutions online at `http://www.dci.pomona.edu/tools-bin/cs081upload.php`. If you have more than one file to be turned in, please put it in a folder and zip it up before turning it in.

Problems from the texts are given in the form c.n where c is the chapter and n is the problem number. Thus problem 2.7 is problem 7 from Chapter 2.

1. (5 points) **Predicate Logic**

   Problem 2.1.3 from H & R page 157.

2. (10 points) **Free & Bound Variables**

   (a) In the formula below, or a copy of it, underline the occurrences of free variables and circle the occurrences of bound variables. (Consider the variable right after a quantifier to be bound.)

   $$\forall z \Big(P(x) \land \forall x \, \exists y \, \big(Q(y, f(z)) \to R(g(g(w,x), f(y))) \big) \Big)$$

   Using the definitions included in the LaTeX file, the command {\encircle x} can be used to draw a letter with a circle around it: x *\underline{x} will generate* $\underline{x}$.

   (b) Let ϕ be the formula in part a. Write out the formulas below, renaming bound variables when necessary to avoid "capture."

   i. $\phi[f(y)/x]$

   ii. $\phi[f(y)/z]$

   iii. $\phi[f(y)/w]$

3. (10 points) **Proofs**

   Exactly one of the inferences below is correct. Give a (constructive) proof of the correct member and a short intuitive explanation of why the other is incorrect.

   (a) $\forall x \, (\phi \lor \psi) \vdash \forall x \, \phi \lor \forall x \, \psi$ and

   (b) $\forall x \, \phi \lor \forall x \, \psi \vdash \forall x \, (\phi \lor \psi)$

4. (10 points) **Bad Proof**

   Recall the axioms for an equivalence relation $\sim$:

   A0. $\forall x \, \big(x \sim x \big)$

   A1. $\forall x \, \forall y \, \big(x \sim y \to y \sim x \big)$

   A2. $\forall x \, \forall y \, \forall z \, \big(x \sim y \to \big(y \sim z \to x \sim z \big) \big)$

   Consider the following proof sketch that purports to prove that $\mathrm{A1}, \mathrm{A2} \vdash \mathrm{A0}$. Formalize the sketch and use your formulation to identify the error in it.

   > Let x be arbitrary and suppose that $x \sim y$. Then by A1, $y \sim x$. Using these two facts, we may use A2 to conclude that $x \sim x$. Because x was arbitrary, we have $\forall x \, \big(x \sim x \big)$.

5. (10 points) **Proofs**

   Problem 2.3.11ac from H & R page 162.

6. (10 points)**Semantics**

   Problem 2.4.5ab from H & R page 163.

7. (10 points) **Consistency**

   Problem 2.4.11ad from H & R page 164. Show each set of formulas is consistent by finding a model for all formulas in the set.

8. (5 points) **Validity**

   Problem 2.5.1d from H & R page 165.

9. (5 points) **Compactness Theorem**

   Consider a predicate logic with a two-place predicate symbol $<$, as well as the usual arithmetic operators $+$ and $*$. Let $\mathbb{N}$ be the model of this language that consists of the natural numbers with their natural ordering and where $+$ and $*$ have their ordinary meanings. Let $\mathrm{Th}(\mathbb{N})$ be the set of all sentences of this language that are true in $\mathbb{N}$. Use the compactness theorem to show that there is another model $\mathfrak{M}$ of this language that satisfies all of the sentences in $\mathrm{Th}(\mathbb{N})$, but also has an infinite descending chain, i.e., an infinite set of elements of the model $\{a_i | i > 0\}$ such that for all $i > 0$, $a_{i+1} <^{\mathfrak{M}} a_i$.