Computability and Logic Homework 1

Due: Thursday, September 8, 2005

Practice Problems (Don't hand in)

Exercises 1.3.8,11,17,18 from the text.[1]
Exercises 3.1.2,4,7,21; 11.1.3,4,7cdf,9 from the text.

Problems To Hand In

Each of problems 1–5 is worth 10 points.

1. Let $\Sigma = \{a, b\}$.

a. Describe the language $\{w \in \Sigma^* \mid ww = www\}$ as simply as you can.

b. Give some examples of strings in, and not in, the language $\{w \in \Sigma^* \mid \textit{for some } u \in \Sigma^*,\ www = uu\}$.

2. Let f be a function from positive integers to nonnegative integers (*i.e.*, $f(0)$ is undefined but $f(i)$ may be 0). In this problem you will work with the set of *chains of* f, defined by:

$$C_f = \{n \geq 0 \mid \textit{for some } k \geq 0,\ f^k(n) = 0\}$$

(We'll consider $f^0(n) = n$ for any n; this means that $0 \in C_f$.)
For example, for the function f depicted below, 0, 4, 57, 23 and 3 are chains; 1 and 2 are not.

f(*n*)		2	1	23	0	...	57	...	4	...
n	0	1	2	3	4	...	23	...	57	...

a. Give an inductive definition of the set C_f.

b. Prove (by induction on k) that if $k \geq 0$ and $f^k(n) = 0$ then $n \in C_f$ according to your inductive definition.

c. Prove by induction that for each n in C_f according to your definition, there is a **unique** k such that $f^k = (0)$. (Remember that $f(0)$ is undefined.)

d. For any chain n, call the unique k whose existence you established in part (c) the *chain length* of n. Give a recursive definition of the function len from chains to natural numbers such that $len(n)$ is the chain length of n.

3. Let $\Sigma = \{a, b\}$. Show that the following language is regular:

$$\{w \in \Sigma^* \mid w \textit{ contains exactly one occurrence of the substring } aaa\}$$

[1] This notation means "exercises 8, 11, 17 and 18 from Section 1.3 of the text." These exercises can be found on pages 52–54.

4. If L is a language, define the *reversal* of L by $L^R = \{w^R \mid w \in L\}$.

a. Prove that for any language L, $(L^*)^R = (L^R)^*$.

b. Prove that if L is a regular language, then so is L^R. You may use part (a).

5.

a. Suppose that L and M are languages with $\Lambda \notin M$. Prove that $L = ML$ if and only if $L = \emptyset$.

b. Prove that if $\Lambda \notin N$ and $L = M \cup NL$ then $L = N^*M$.

c. Prove that if

$$\begin{aligned} L_1 &= M_0 \cup M_1 L_1 \cup M_2 L_2 \\ \textit{and} \quad L_2 &= N_0 \cup N_1 L_1 \cup N_2 L_2 \end{aligned}$$

and all the M_i's and N_i's are regular and $\Lambda \notin (M_1 \cup M_2 \cup N_1 \cup N_2)$, then L_1 and L_2 are regular. You may use part (b) and any result from the practice problems.

d. What happens if we allow N to contain Λ in part (b)? How about M_1 in part (c)? How about M_2?