Math 60
A brief primer on complex arithmetic

What are the complex numbers? Starting with a field of play among real numbers, \( \mathbb{R} \), we observe that certain polynomials like \( x^2 + 1 \) do not have roots, since there are no real numbers \( r \) with the property that \( r^2 + 1 = 0 \), or \( r^2 = -1 \). Being intrepid math explorers (xplorers?), we’re not going to let that stop us. So we expand our field of play to contain the real numbers as well as solutions to polynomials we could not otherwise solve. To that end, we define \( i = \sqrt{-1} \), and we insist on maintaining the same ability to add, subtract, multiply, and divide. So for instance, since 47 and \( i \) are in our field of play, so must be 47 + \( i \) and 47\( i \). Thus we define the complex numbers to be the set

\[ \mathbb{C} = \{ a + bi \mid a, b \in \mathbb{R} \} \]

According to this definition, 47 + \( i \), 47\( i \) \( \in \mathbb{C} \). Addition and multiplication follow the standard rules. For instance, \((3 + 2i) + (1 - i) = 4 + i\), and \((3 + 2i)(1 - i) = 3 - 3i + 2i - 2i^2 = 3 - 3i + 2i + 2 = 5 - i\). Also, from the definition we see that \( \mathbb{R} \subset \mathbb{C} \), since \( \mathbb{R} = \{ a + bi \mid a, b \in \mathbb{R} \text{ and } b = 0 \} \).

But what about division? That requires a teensy bit more care. First, it helps to recall something you may have learned in kindergarten: how to rationalize a denominator. For instance, the expression \( \frac{1}{3 + 2\sqrt{5}} \) displeases our aesthetic sensibilities. So we make use of the property \( (a + bi)(a - bi) = a^2 - b^2 \) to get rid of the square root in the denominator:

\[
\frac{1}{3 + 2\sqrt{5}} = \frac{1}{3 + 2\sqrt{5}} \left( \frac{3 - 2\sqrt{5}}{3 - 2\sqrt{5}} \right) = \frac{3 - 2\sqrt{5}}{3^2 - (2\sqrt{5})^2} = \frac{3 - 2\sqrt{5}}{9 - 20} = \frac{3 - 2\sqrt{5}}{-11} = -\frac{3}{11} + \frac{2}{11}\sqrt{5}
\]

Much better! We can use the same trick to compute the reciprocal of a complex number and write it in standard form (recalling as before that \( i^2 = -1 \)). For instance:

\[
\frac{1}{3 + 2i} = \frac{1}{3 + 2i} \left( \frac{3 - 2i}{3 - 2i} \right) = \frac{3 - 2i}{3^2 - (2i)^2} = \frac{3 - 2i}{13} = \frac{3}{13} - \frac{2}{13}i
\]

Armed with reciprocals, we can divide complex numbers fearlessly:

\[
\frac{1 - i}{3 + 2i} = (1 - i) \left( \frac{3}{13} - \frac{2}{13}i \right) = \frac{3}{13} - \frac{2}{13}i - \frac{3}{13}i + \frac{2}{13}i^2 = \frac{1}{13} - \frac{5}{13}i
\]

And armed with division (and multiplication and subtraction and addition), we can perform row reduction on systems of linear equations with complex coefficients, like, say, problem 27 of chapter 6.

Is that it for the complex numbers? As far as our class is concerned, yes, more or less. But the complex numbers are amazing in their own right. Here are three tastes of \( \mathbb{C} \) to whet the appetite.

The Fundamental Theorem of Arithmetic: We constructed the complex numbers to that polynomials like \( x^2 + 1 \) will have roots we can play with. But we got more than we asked for in that bargain. A lot more. In fact, every polynomial of degree \( n \) with coefficients in the complex numbers (this includes every polynomial with real coefficients since \( \mathbb{R} \subset \mathbb{C} \)) has all \( n \) of its roots in the complex numbers. Every one! We have found a field of play where every root of every polynomial lives! The proof relies, in part, on the fact that \( \mathbb{C} \) is a vector space over \( \mathbb{R} \) of dimension 2 (with basis \( \{ 1, i \} \) ). To learn more about this, take Math 172, Galois Theory.

Liouville’s Theorem: Suppose \( f(x) \) is a function from \( \mathbb{C} \) to \( \mathbb{C} \) with the following properties: \( f \) has to be differentiable and bounded. You can probably think of many such functions over the real numbers. But over \( \mathbb{C} \), the only such functions are constants! To repeat—because this is so cool—if \( f : \mathbb{C} \to \mathbb{C} \) is differentiable and bounded, it must be constant. Take Math 135, Functions of a Complex Variable, to learn more about this.

The Riemann Hypothesis: Consider the infinite series \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \). From calculus I, we know that \( \zeta(s) \) converges when \( s > 1 \) and diverges when \( s \leq 1 \). It turns out that we can shed that restriction on \( s \) by defining \( \zeta(s) \) in a way that makes sense for any \( s \in \mathbb{C} \) via a technique called analytic continuation. Then we ask: for which \( s \in \mathbb{C} \) does \( \zeta(s) = 0 \)? It is known that \( \zeta(-2) = \zeta(-4) = \zeta(-6) = \cdots = 0 \). It is conjectured that, aside from the negative even cases, if \( \zeta(s) = 0 \) then \( s = \frac{1}{2} + bi \). Prove of disprove and win a million dollars.