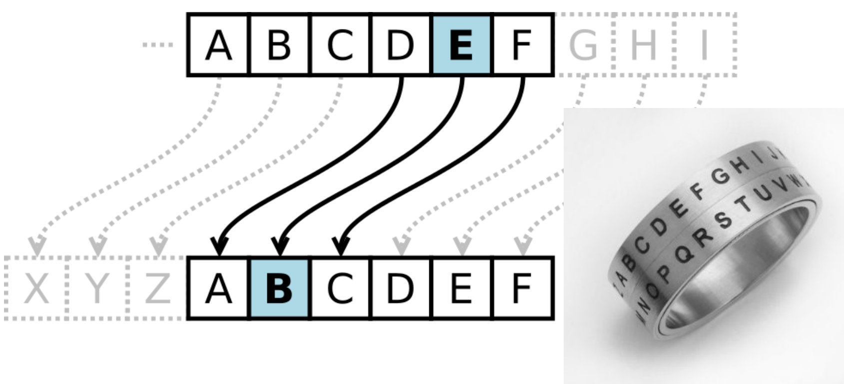
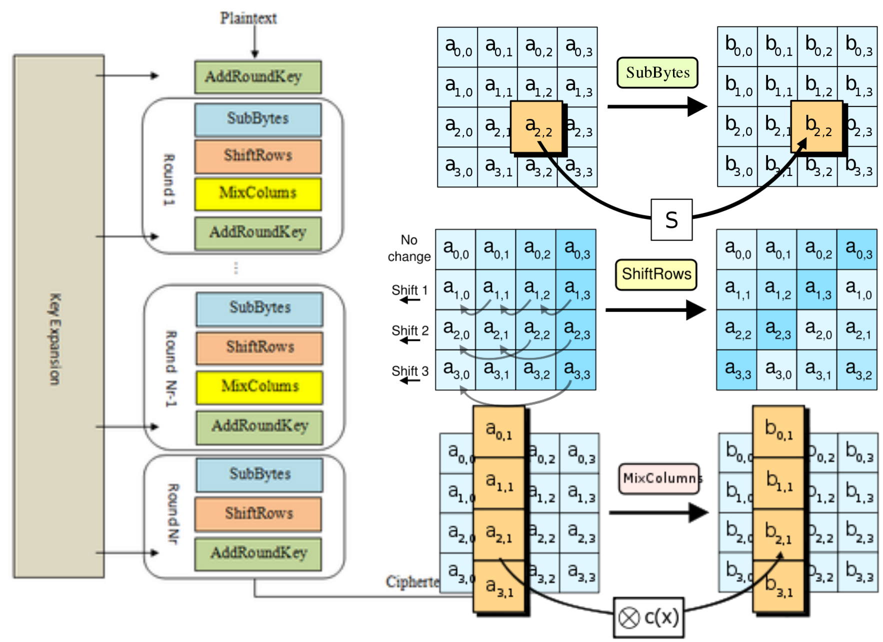
Lecture 7: Public-Key Cryptography

CS 181S

Fall 2020

Crypto Thus Far…



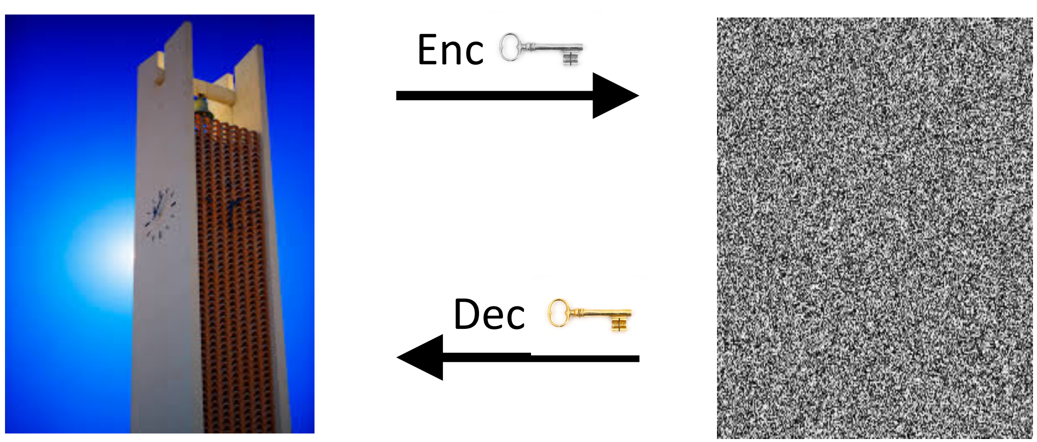


Key pairs

- Instead of sharing a key between pairs of principals...
- ...every principal has a pair of keys
  - **public key:** published for the world to see
  - **private key:** kept secret and never shared

(Public-Key) Encryption algorithms

- $\mathrm{Gen}(1^n)$: generate a **keypair** (pk, sk) of length n
- $\mathrm{Enc}(m; pk)$: encrypt **message** under public key pk
- $\mathrm{Dec}(c; sk)$: decrypt **ciphertext** c with secret key sk



$(\mathrm{Gen}, \mathrm{Enc}, \mathrm{Dec})$ is a public-key **encryption scheme** aka **cryptosystem**

RSA

[Rivest, Shamir, Adleman 1977]

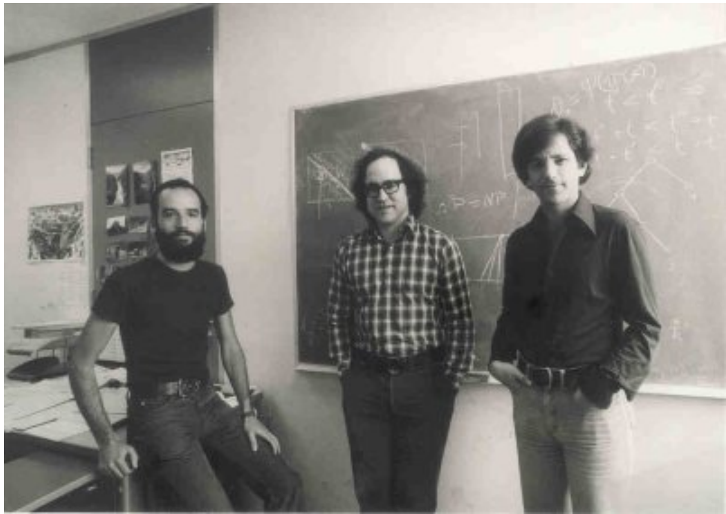
Shared Turing Award in 2002: *ingenious contribution to making public-key crypto*

- Gen(len):
  - Pick primes p, q, define $n = p \cdot q$
  - Choose e, d such that $ed = 1 \bmod (p-1)(q-1)$
  - $pk = (n, e), \;\; sk = (p, q, d)$
- Enc(m, pk)

$$c = m^e \bmod n$$

- Dec(c, sk):

$$m = c^d \bmod n$$



Exercise 1: RSA

- Let $pk = (n, e) = (21, 5)$ and $sk = (p, q, d) = (3, 7, 5)$
- Observe that $ed = 5 \cdot 5 = 25 = 1 \bmod 12$

1. Compute $c = \mathrm{Enc}(17; pk)$

   $$c = \mathrm{Enc}\big(17; (21,5)\big) = 17^5 \bmod 21 = 1419857 \bmod 21 = 5$$

2. Compute $\mathrm{Dec}(c; sk)$

   $$\mathrm{Dec}(c; sk) = \mathrm{Dec}\big(5; (3,7,5)\big) = 5^5 \bmod 21 = 3125 \bmod 21 = 17$$

RSA

- Theorem: RSA is a correct public-key encryption scheme.
- Theorem: $(m^e \bmod pq)^d \bmod pq == m$

$$
\begin{aligned}
\mathrm{Dec}(\mathrm{Enc}(m; pk); sk) &= \ (m^e \bmod pq)^d \bmod pq \\
&= \ (m^e)^d \ \bmod pq \\
&= \ m^{ed} \bmod pq \\
&= m \bmod pq
\end{aligned}
$$

$$
\begin{aligned}
m^{ed} \bmod p &= m^{1+k(p-1)(q-1)} \bmod p \\
&= m \cdot (m^{p-1})^{\ k(q-1)} \ \bmod p \\
&= \ m \cdot (m^{p-1} \bmod p)^{\ k(q-1)} \ \bmod p \\
&= m \cdot (1)^{\ k(q-1)} \ \bmod p \\
&= m \bmod p
\end{aligned}
$$

$$
\begin{aligned}
m^{ed} \bmod q &= m^{1+k(p-1)(q-1)} \bmod q \\
&= m \cdot (m^{q-1})^{\ k(p-1)} \ \bmod q \\
&= \ m \cdot (m^{q-1} \bmod p)^{\ k(p-1)} \ \bmod q \\
&= m \cdot (1)^{\ k(p-1)} \ \bmod q \\
&= m \bmod q
\end{aligned}
$$

RSA

- ~~**Theorem:** RSA is a secure public-key encryption scheme.~~

- Rabin Encryption (integer factorization)
- ElGamal Encryption (discrete log)
- Pailler Encryption (composite residuosity)
- Elliptic Curve Integrated Encryption Scheme (comp. DH)

Problems with Textbook RSA

- *Deterministic*: given same plaintext and key, always produces the same ciphertext

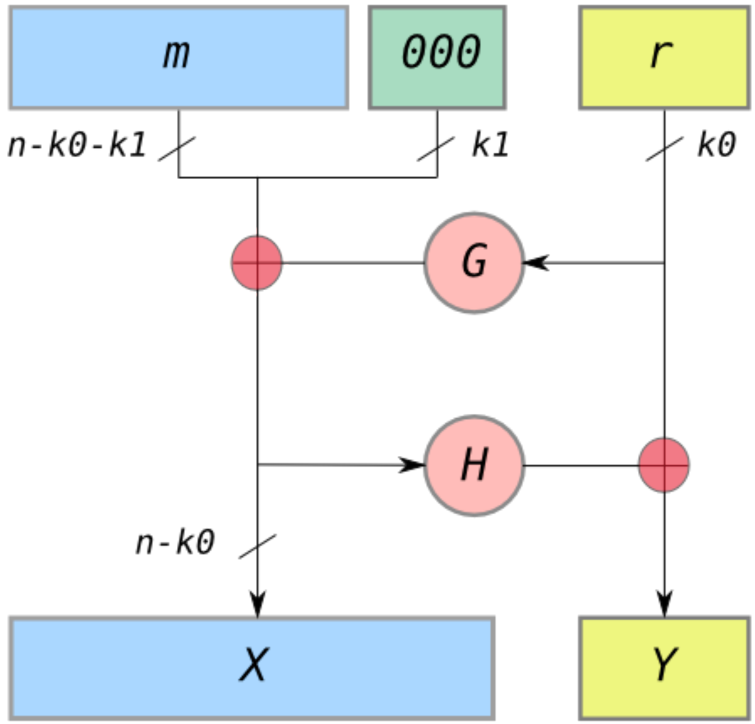
- *Small numbers*: if m^e < n, then log is easy to compute
- *Big numbers*: if m > n, can't compute do math mod n

- *Side channel attacks*: interfaces can leak information about secret key
- *Key Management:* no secure place to store the secret key

- *Quantum Computers*: provably breakable with different hardware

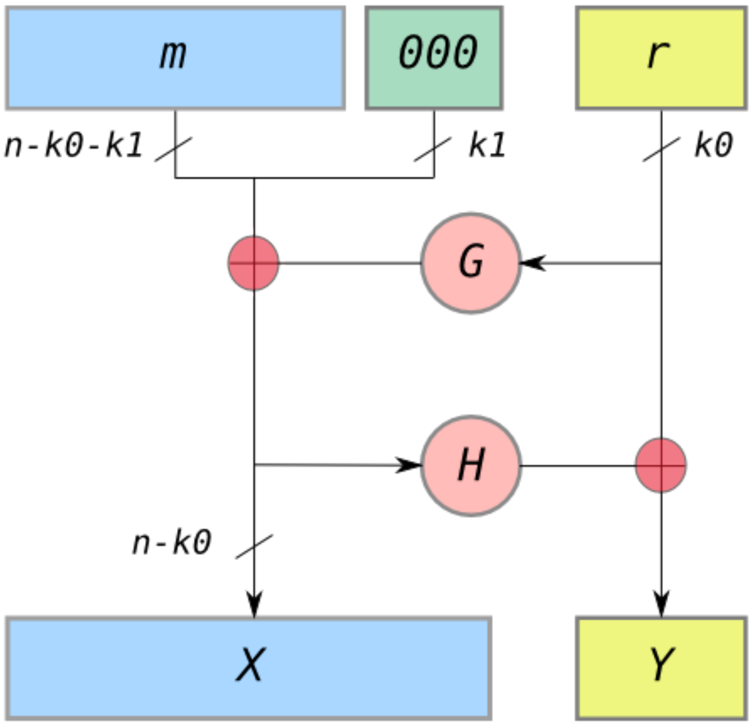
Solution 1: Padding

- PKCS#1 v1.5: 0x00 0x02 [non-zero bytes] 0x00 [message]
  - Vulnerable to a padding oracle attack!
- OAEP (Optimal Asymmetric Encryption Padding)
  - Security proof (with assumptions)



Exercise 2: OAEP

- Define a function to compute m given values X and Y, constants k0 and k1, and hash functions G and H

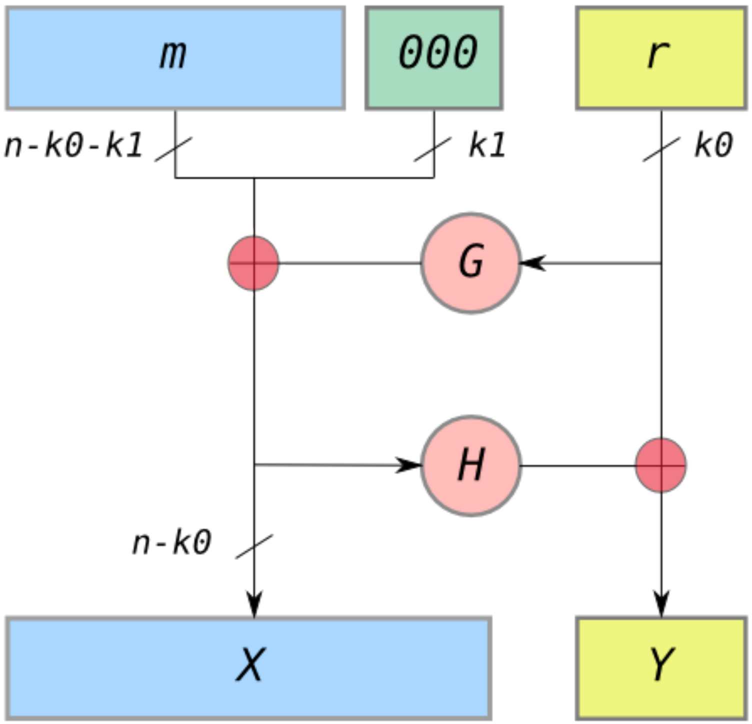


Exercise 2: OAEP

- Define a function to compute m given values X and Y, constants k0 and k1, and hash functions G and H

```
extract_m(X, Y){
  r = H(X)^Y;
  m' = X^G(r);
  m = m' >> k1;

  return m;
}
```



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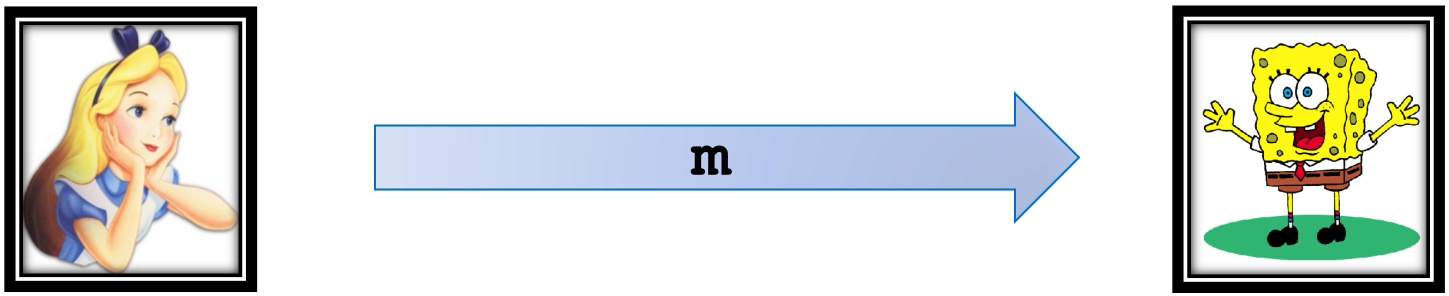
Solution 2: Hybrid encryption

- Assume:
  - Symmetric encryption scheme (Gen_SE, Enc_SE, Dec_SE)
  - Public-key encryption scheme (Gen_PKE, Enc_PKE, Dec_PKE)

- Use public-key encryption to establish a shared session key
  - Avoids quadratic problem, assuming existence of phonebook
  - Avoids problem of key distribution

- Use symmetric encryption to exchange long plaintext encrypted under session key
  - Gain efficiency of block cipher and mode

Protocol to exchange encrypted message




```
0.   B: (pk_B, sk_B) = Gen_PKE(len_PKE)
        publish (B, pk_B)
1.   A: k_s = Gen_SE(len_SE)
        c1 = Enc_PKE(k_s; pk_B)
        c2 = Enc_SE(m; k_s)
2.   A -> B: c1, c2
3.   B: k_s = Dec_PKE(c1; sk_B)
        m = Dec_SE(c2; k_s)
```

Session keys

- If key compromised, only those messages encrypted under it are disclosed
- Used for a brief period then discarded
  - cryptoperiod*:* length of time for which key is valid
  - in this case, for a single (long) message
  - not intended for reuse in future messages
- only intended for unidirectional usage:
    - A->B, not B->A

Problems with Textbook RSA

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Square-and-Multiply

```
int modular_exp(x, n, p){
    res = 1;
    while (n > 0) {
        if (n % 2 == 1){
            res = res * x % p;
        }
        x = x^2 % p;
        n >> 1;
    }
    return res;
}
```

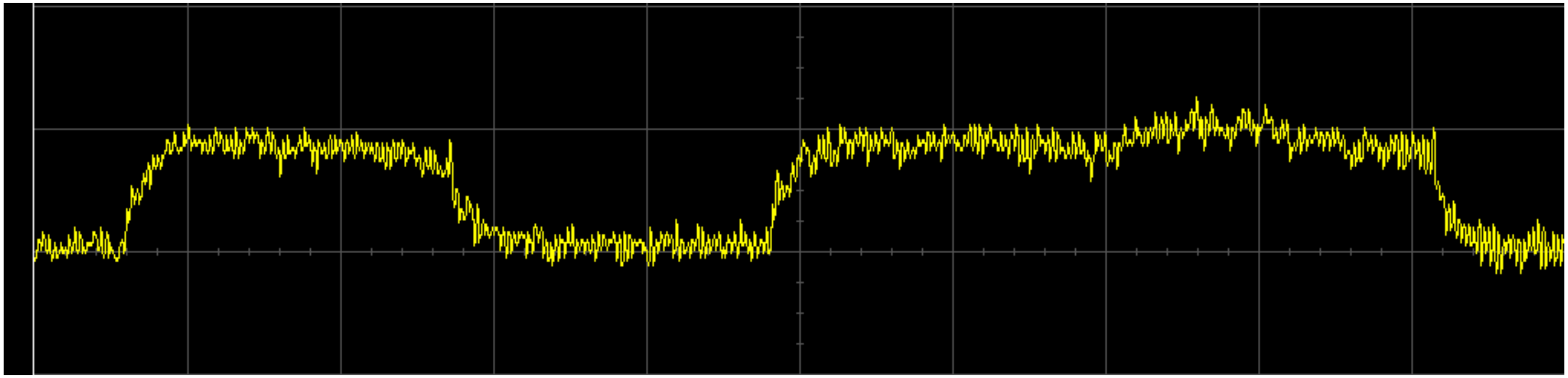
Exercise 3: Square-and Multiply

- Compute $3^5 \bmod 21$ using square and multiply

```
res = 1      x = 3      n = 5
res = 3      x = 9      n = 2
res = 3      x = 18     n = 1
res = 12     x = 9      n = 0
```

```
int modular_exp(x, n, p){
   res = 1;
   while (n > 0) {
       if (n % 2 == 1){
          res = res * x % p;
       }
       x = x^2 % p;
       n >> 1;
   }
   return res;
}
```

Side Channels



- Power
- Timing
- EM Radiation
- Acoustics

Solution 3: Blinded RSA

[Rivest, Shamir, Adleman 1977]

Shared Turing Award in 2002: *ingenious contribution to making public-key crypto*

- Gen(len):
  - Pick primes p, q
  - Choose e, d such that $ed = 1 \bmod \operatorname{lcm}(p-1, q-1)$
  - $pk = (n, e), \quad sk = (p, q, d)$
- Enc(m, pk)

$$c = m^e \bmod n$$

- Dec(c, sk):

$$m = ((cr)^d \bmod n) \cdot r^{-d}$$

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Solution 4: Key Management

- Store keys offline
- Store keys in protected files
- Memorize the keys (sort of)

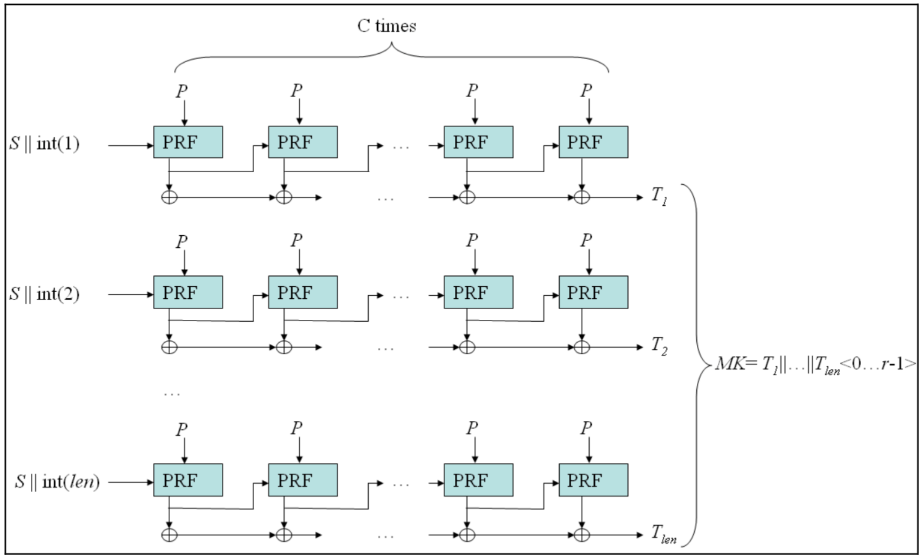
Password-Based Encryption

- PBKDF2: Password-based key derivation function [RFC 8018]
- **Output:** derived key k
- **Input:**
  - Password p
  - Salt s
  - Iteration count c
  - Key length len
  - Pseudorandom function (PRF): "looks random" to an adversary that doesn't know an input called the *seed* (commony instantiated with an HMAC)

PBKDF2

Algorithm:

- F(p, s, i, c) = U(1) XOR ... XOR U(c)
  - U(1) = PRF(s, i; p)
  - U(j) = PRF(U(j-1); p)
  - F is in essence a salted iterated hash...
- k = F(p, s, 1, c) || F(p, s, 2, c) || ... || F(p, s, n, c)
  - enough copies to reach keylen
  - || denotes bit concatenation



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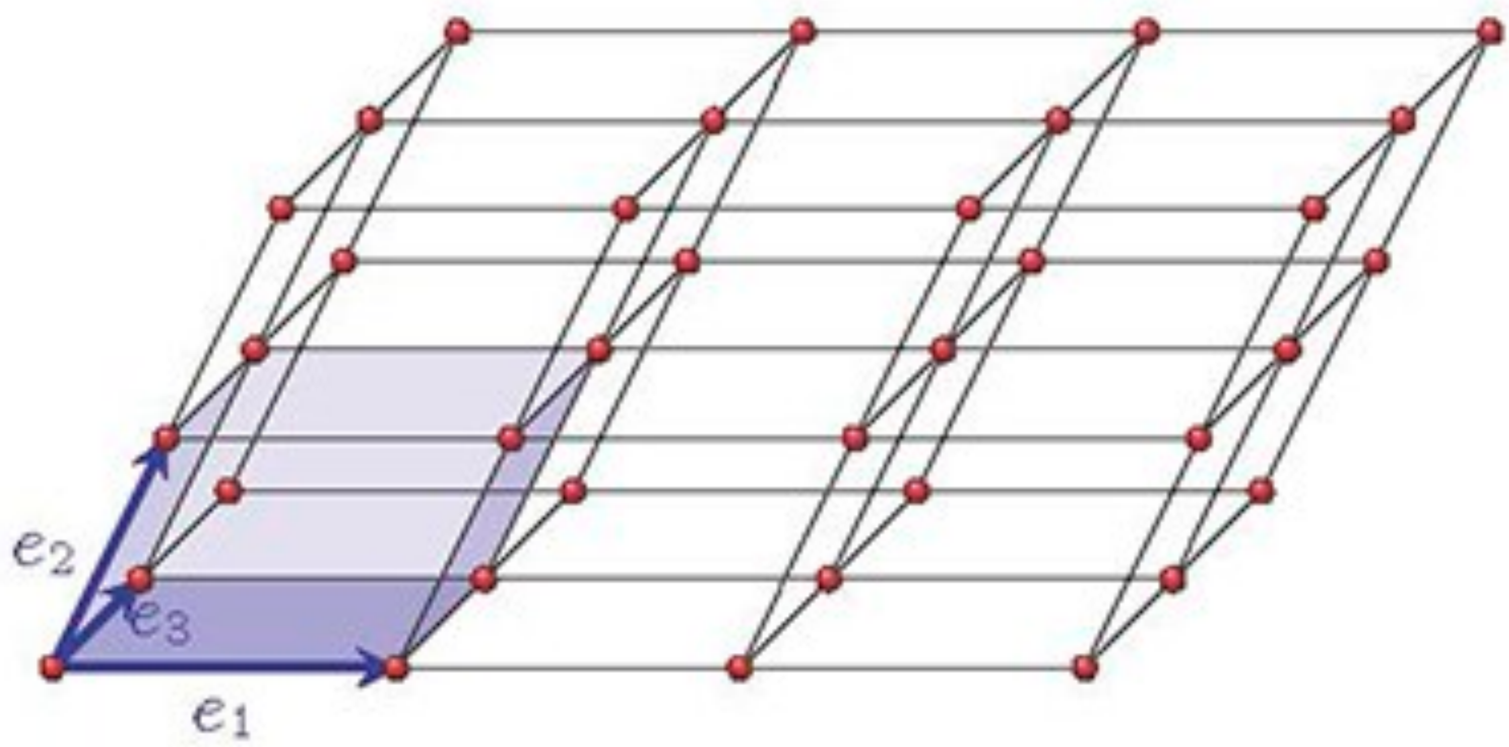
Solution 5: Post-Quantum Cryptography



Exercise 4: Feedback

1. Rate how well you think this recorded lecture worked
   1. Better than an in-person class
   2. About as well as an in-person class
   3. Less well than an in-person class, but you still learned something
   4. Total waste of time, you didn't learn anything

2. How much time did you spend on this video lecture (including time spent on exercises)?

3. Do you have particular questions you would like me to address in this week's problem session?

4. Would you prefer to keep using this asynchronous/flipped classroom approach or would you prefer to switch to synchronous teaching?

5. Do you have any other comments or feedback?