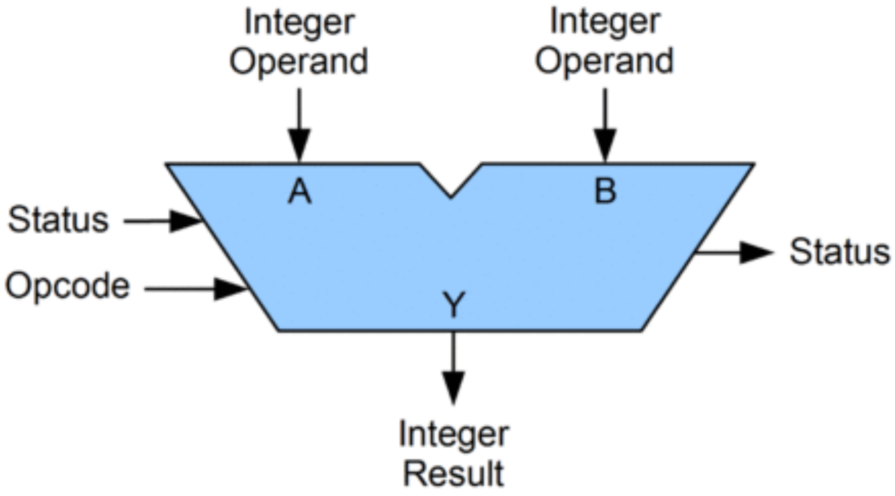
Lecture 4: Operations on Values

CS 105

February 3, 2020

Arithmetic Logic Unit (ALU)

- circuit that performs bitwise operations and arithmetic on integer binary types



Boolean Algebra

- Developed by George Boole in 19th Century
- Algebraic representation of logic---encode “True” as 1 and “False” as 0

And

&	0	1
0	0	0
1	0	1

Or

|	0	1
0	0	1
1	1	1

Not

~	
0	1
1	0

Exclusive-Or (Xor)

^	0	1
0	0	1
1	1	0

General Boolean algebras

- Bitwise operations on words

```
  01101001     01101001     01101001
& 01010101   | 01010101   ^ 01010101   ~ 01010101
  01000001     01111101     00111100     10101010
```

- How does this map to set operations?

Exercise: Boolean algebras

- Assume: `a = 01101101, b = 01010101`

- What are the results of evaluating the following Boolean operations?

  - `~a`
  - `~b`
  - `a & b`
  - `a | b`
  - `a ^ b`
  - `((a ^ b) & ~b) | (~(a ^ b) & b)`

Example: Using Boolean Operations

```
void f(int *x, int*y){
    *y = *x ^ *y;
    *x = *x ^ *y;
    *y = *x ^ *y;
}
```

- What does this function do?

Bitwise vs Logical Operations in C

- Apply to any "integral" data type
  - `int, unsigned, long, short, char`
- Bitwise Operators &, |, ~, ^
  - View arguments as bit vectors
  - operations applied bit-wise in parallel
- Logical Operators `&&, ||, !`
  - View 0 as "False"
  - View anything nonzero as "True"
  - Always return 0 or 1
  - Early termination

Exercise: Bitwise vs Logical Operations

- Assume char data type (one byte)

  - `~0x41`
  - `~0x00`
  - `~~0x41`

  - `0x69 & 0x55`
  - `0x69 | 0x55`

  - `!0x41`
  - `!0x00`
  - `!!0x41`

  - `0x69 && 0x55`
  - `0x69 || 0x55`

Bit Shifting

- Left Shift: **`x << y`**
  - Shift bit-vector **`x`** left **`y`** positions
  - Throw away extra bits on left
  - Fill with 0's on right

Undefined Behavior if you shift amount < 0 or ≥ word size

- Right Shift: **`x >> y`**
  - Shift bit-vector **`x`** right **`y`** positions
  - Throw away extra bits on right
  - Logical shift: Fill with 0's on left
  - Arithmetic shift: Replicate most significant bit on left

Choice between logical and arithmetic depends on the type of data

Example: Bit Shifting

- `Unsigned`
  - `0x41 << 4`
  - `0x41 >> 4`

- `Signed`
  - `41 << 4`
  - `41 >> 4`
  - `-41 << 4`
  - `-41 >> 4`

Addition Example

- Compute 5 + 1 assuming all ints are stored as three-bit unsigned values

- Compute -3 + 1 assuming all ints are stored as three-bit signed values (two's complement)

Addition and Subtraction

- Usual addition and subtraction
    - Like you learned in second grade, only binary
    - Same for unsigned and signed
    - … but error conditions differ

Error Cases

- Unsigned addition:
  - $x +_w^u y = \begin{cases} x + y & \text{(normal)} \\ x + y - 2^w & \text{(overflow)} \end{cases}$
  - overflow has occurred iff $x +_w^u y < x$
- Signed addition:
  - $x +_w^t y = \begin{cases} x + y - 2^w & \text{(positive overflow)} \\ x + y & \text{(normal)} \\ x + y + 2^w & \text{(negative overflow)} \end{cases}$
  - overflow has occurred iff $x > 0$ and $y > 0$ and $x +_w^t y < 0$
    or $x < 0$ and $y < 0$ and $x +_w^t y > 0$

Flags

- A flag is a one-bit value: 1 is “set” and 0 is “unset”
- Flags record conditions of previous arithmetic operations

  - **C**: The carry-out flag from the last bit; indicates unsigned overflow
  - **V**: Indicates if the result, interpreted as a signed value, is erroneous. For addition, this means that the signs of the operands agree and the result has a different sign
  - **Z**: Set if the result is zero
  - **N**: The sign bit of the result; indicates a negative signed result

Multiplication Example

- Compute 5 * 3 assuming all ints are stored as three-bit unsigned values

- Compute -3 * 3 assuming all ints are stored as three-bit signed values (two's complement)

Multiplication

- Usual Multiplication
    - Like elementary school, only in binary
    - Product can be two words long; it may be truncated to one word
    - Bit level equivalence for unsigned and signed

Error Cases

- Unsigned multiplication:
  - $x *_w^u y = (x \cdot y) \bmod 2^w$

- Signed multiplication:
  - $x *_w^t y = U2T((x \cdot y) \bmod 2^w)$

Multiplying with Shifts

C uses << and >>. The arithmetic/logical choice is made according the the operands being signed/unsigned.

Java has no unsigned integers, but it has a third shift >>> for logical right shift.

We can multiply (often faster than with the processor's multiply instruction) with shifts.

- $$\begin{aligned} x \times 24 &= x \times 32 - x \times 8 \\ &= (x << 5) - (x << 3) \end{aligned}$$

Most compilers will generate this code automatically.

Signed Division by a Power of 2

- `x >> k` computes `x / 2`^{k}, rounded towards $-\infty$
- C on Intel processors rounds towards 0
  - `-11 >> 2 == -3`, but `-11/4 == -2`
- Solution: If x < 0, add 2^k-1 before shifting
  - Why does this work?

```
if (x < 0)
   x += (1 << k) - 1;
return x >> k;
```