INDUCTION

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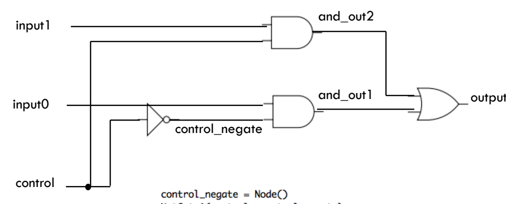
CS52 – Spring 2016

2-to-1 multiplexer




```
control_negate = Node()
NotGate1(control, control_negate)

and_out1 = Node()
AndGate2(input0, control_negate, and_out1)

and_out2 = Node()
AndGate2(input1, control, and_out2)

OrGate2(and_out1, and_out2, output)
```

A useful identity

What is the sum of the powers of 2 from from 0 to n?

$$\sum_{i=0}^{n} 2^i = 2^0 + 2^1 + \ldots + 2^n = \text{?}$$

A useful identity

The sum of the powers of 2 from from 0 to n is:

$$\sum_{i=0}^{n} 2^i = 2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$$

For example, what is: $\sum_{i=0}^{4} 2^i$? $\sum_{i=0}^{9} 2^i$?

$1 + 2 + 4 + 8 + 16 = 31 = 2^5\text{-}1$

$1 + 2 + 4 + 8 + \ldots + 2^9 = 2^{10} - 1 = 1023$

A useful identity

The sum of the powers of 2 from from 0 to n is:

$$\sum_{i=0}^{n} 2^i = 2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$$

How would you **prove** this?

Proof by induction

1. State what you're trying to prove!
2. State and prove the base case
   - What is the smallest possible case you need to consider?
   - Should be fairly easy to prove
3. Assume it's true for k (or k-1). Write out specifically what this assumption is (called the *inductive hypothesis*).
4. Prove that it then holds for k+1 (or k)
   a. State what you're trying to prove (should be a variation on step 1)
   b. Prove it. You will need to use inductive hypothesis.

An example

1. State what you're trying to prove!

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

An example

1. $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

2. Base case:

   What is the smallest possible case you need to consider?

An example

1. $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

2. Base case:

n = 0

What does the identity say the answer should be?

An example

1. $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

2. Base case:

n = 0

$$\sum_{i=0}^{0} 2^i = 2^{0+1} - 1 = 1$$

Is that right?

An example

1. $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

2. Base case:

n = 0

$$\sum_{i=0}^{0} 2^i = 2^{0+1} - 1 = 1$$

$$\updownarrow$$

Base case proved!

$$\sum_{i=0}^{0} 2^i = 2^0 = 1$$

An example

1. $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

3. Assume it's true for some k *(inductive hypothesis)*

Assume: $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

4. Prove that it's true for k+1

   a. State what you're trying to prove:

$$\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$$

Prove it!

Assuming: $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$ Prove: $\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$

$$\sum_{i=0}^{k+1} 2^i = 2^0 + 2^1 + \ldots + 2^k + 2^{k+1}$$

Ideas?

Prove it!

Assuming: $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$ Prove: $\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$

$$\sum_{i=0}^{k+1} 2^i = \underbrace{2^0 + 2^1 + \ldots + 2^k}_{} + 2^{k+1}$$

$= 2^{k+1} - 1 + 2^{k+1}$ by the inductive hypothesis

$= 2 * 2^{k+1} - 1$ by math (combine the two 2^{k+1})

$= 2^{k+2} - 1$ by math

Done!

Proof like I'd like to see it on paper (part 1)

1. Prove: $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

2. Base case: n = 0

   Prove: $\sum_{i=0}^{0} 2^i = 2^{0+1} - 1$

   LHS: $\sum_{i=0}^{0} 2^i = 2^0 = 1$ by math

   RHS: $2^{0+1} - 1 = 2 - 1 = 1$ by math

   LHS = RHS

Proof like I'd like to see it on paper (part 2)

3. Assuming it's true for n = k, i.e.

$$\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$$

4. Show that it holds for k+1, i.e.

$$\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$$

Proof like I'd like to see it on paper (part 3)

$$\sum_{i=0}^{k+1} 2^i = 2^0 + 2^1 + \ldots + 2^k + 2^{k+1}$$

$= 2^{k+1} - 1 + 2^{k+1}$ by the inductive hypothesis

$= 2 * 2^{k+1} - 1$ by math (combine the two 2^{k+1})

$= 2^{k+2} - 1$ by math

Done!

Proof by induction

1. State what you're trying to prove!
2. State and prove the base case
3. Assume it's true for k (or k-1)
4. Show that it holds for k+1 (or k)

Why does this prove anything?

Proof by induction

We proved the base case is true, e.g. $\sum_{i=0}^{0} 2^i = 2^1 - 1$

If k = 0 is true (the base case) then k = 1 is true $\sum_{i=0}^{1} 2^i = 2^2 - 1$

If k = 1 is true then k = 2 is true $\sum_{i=0}^{2} 2^i = 2^3 - 1$

…

If n-1 is true then n is true $\sum_{i=0}^{n} 2^i = 2^n - 1$

Another useful identity

What is the sum of the numbers from 1 to n?

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \text{ ?}$$

A useful identity

The sum of the numbers from 1 to n is:

$$\sum_{i=1}^{n} i = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$$

For example, what is sum from 1 to 5? 1 to 100?

1 + 2 + 3 + 4 + 5 = 15 = 5*6/2

1 + 2 + 3 + … + 100 = 100 * 101/2 = 10100/2 = 5050

Prove it!

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

1. State what you're trying to prove!
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   - What is the smallest possible case you need to consider?
   - Should be fairly easy to prove
3. Assume it's true for k (or k-1). Write out specifically what this assumption is (called the *inductive hypothesis*).
4. Prove that it holds for k+1 (or k)
   a. State what you're trying to prove (should be a variation on step 1)
   b. Prove it. You will need to use the inductive hypothesis.