A useful identity

What is the sum of the powers of 2 from 0 to n?

\[ \sum_{i=0}^{n} 2^i = 2^0 + 2^1 + \ldots + 2^n = ? \]

The sum of the powers of 2 from 0 to n is:

\[ \sum_{i=0}^{n} 2^i = 2^{i+1} - 1 \]

For example, what is: \( \sum_{i=0}^{4} 2^i \)? \( \sum_{i=0}^{3} 2^i \)?

- \( 1 + 2 + 4 + 8 + 16 = 31 = 2^5 - 1 \)
- \( 1 + 2 + 4 + 8 + \ldots + 2^9 = 2^{10} - 1 = 1023 \)
A useful identity

The sum of the powers of 2 from 0 to n is:

\[ \sum_{i=0}^{n} 2^i = 2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1 \]

How would you prove this?

Proof by induction

1. State what you’re trying to prove!
2. State and prove the base case
   - What is the smallest possible case you need to consider?
   - Should be fairly easy to prove
3. Assume it’s true for k (or k-1). Write out specifically what this assumption is (called the inductive hypothesis).
4. Prove that it then holds for k+1 (or k)
   a. State what you’re trying to prove (should be a variation on step 1)
   b. Prove it. You will need to use inductive hypothesis.

An example

1. State what you’re trying to prove!
   \[ \sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \]
2. Base case:
   - What is the smallest possible case you need to consider?
   -
An example

1. \( \sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \)

2. Base case:
   \( n = 0 \)
   
   What does the identity say the answer should be?

3. Assume it's true for some \( k \) (inductive hypothesis)
   
   Assume \( \sum_{i=0}^{k} 2^i = 2^{k+1} - 1 \)

4. Prove that it's true for \( k+1 \)
   a. State what you're trying to prove:
   
   \[ \sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1 \]
Prove it!

Assuming: \[ \sum_{i=0}^{k} 2^i = 2^{k+1} - 1 \]
Prove: \[ \sum_{i=0}^{k} 2^i = 2^{k+1} - 1 \]

\[ \sum_{i=0}^{k} 2^i = 2^0 + 2^1 + \ldots + 2^k \]

Ideas?

Proof like I'd like to see it on paper (part 1)

1. Prove: \[ \sum_{i=0}^{k} 2^i = 2^{k+1} - 1 \]

2. Base case: \( n = 0 \)
   Prove: \[ \sum_{i=0}^{0} 2^i = 2^{1} - 1 \]
   \[ \text{LHS: } \sum_{i=0}^{0} 2^i = 2^0 = 1 \text{ by math} \]
   \[ \text{RHS: } 2^1 - 1 = 1 \text{ by math} \]
   \[ \text{LHS } = \text{ RHS} \]

Proof like I'd like to see it on paper (part 2)

3. Assuming it's true for \( n = k \), i.e.
   \[ \sum_{i=0}^{k} 2^i = 2^{k+1} - 1 \]

4. Show that it holds for \( k+1 \), i.e.
   \[ \sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1 \]
Proof by induction

1. State what you’re trying to prove!
2. State and prove the base case
3. Assume it’s true for k (or k-1)
4. Show that it holds for k+1 (or k)

Why does this prove anything?

Another useful identity

What is the sum of the numbers from 1 to n?

\[ \sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = ? \]

Proof by induction

We proved the base case is true, e.g. \( \sum_{i=1}^{2} 2^i = 2^1 - 1 \)

If \( k = 0 \) is true (the base case) then \( k = 1 \) is true \( \sum_{i=1}^{2} 2^i = 2^1 - 1 \)

If \( k = 1 \) is true then \( k = 2 \) is true \( \sum_{i=1}^{3} 2^i = 2^2 - 1 \)

…

If \( n-1 \) is true then \( n \) is true \( \sum_{i=1}^{n} 2^i = 2^n - 1 \)
A useful identity

The sum of the numbers from 1 to n is:

\[ \sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \]

For example, what is sum from 1 to 5? 1 to 100?

1 + 2 + 3 + 4 + 5 = 15 = 5*6/2
1 + 2 + 3 + \ldots + 100 = 100 * 101/2 = 10100/2 = 5050

Prove it!

\[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \]

1. State what you’re trying to prove!
2. State and prove the base case
   a. What is the smallest possible case you need to consider?
   b. Should be fairly easy to prove
3. Assume it’s true for k (or k-1). Write out specifically what this assumption is (called the inductive hypothesis).
4. Prove that it holds for k+1 (or k)
   a. State what you’re trying to prove (should be a variation on step 1)
   b. Prove it. You will need to use the inductive hypothesis.