Constraint Satisfaction Problems (CSPs)

CS311
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Quick search recap

- Search
  - uninformed
    - BFS, DFS, IDS
  - informed
    - A*, IDA*, greedy-search
- Adversarial search
  - assume player makes the optimal move
  - minimax and alpha-beta pruning
- Local search (aka state space search)
  - start random, make small changes
  - dealing with local minima, plateaus, etc.
    - random restart, randomization in the approach, simulated annealing, beam search, genetic algorithms

Intro Example: 8-Queens

Where should I put the queens in columns 3 and 4?

Admin

- Final project comments:
  - Use pre-existing code
  - Get your data now!
    - Use pre-existing data sets
  - Finding good references
    - Google scholar (http://scholar.google.com)
    - Other papers
- Final project proposals due tomorrow at 6pm
- Some written problems will be posted tomorrow
Intro Example: 8-Queens

The decisions you make constrain the possible set of next states

Sudoku

What value?
We could try and solve this by searching, but the problem constraints may direct us better allowing for a much faster solution finding.

Constraint satisfaction problem

Another form of search (more or less):

- Set of variables: $x_1, x_2, \ldots, x_n$
- Domain for each variable indicating possible values: $D_{x_1}, D_{x_2}, \ldots, D_{x_n}$
- Set of constraints: $C_1, C_2, \ldots, C_m$
  - Each constraint limits the values the variables can take
    * $x_1 \neq x_2$
    * $x_1 < x_2$
    * $x_1 + x_2 = x_3$
    * $x_4 < x_5$

Goal: an assignment of values to the variables that satisfies all of the constraints

Applications?

Applications

Scheduling:

<table>
<thead>
<tr>
<th>Me</th>
<th>I'd like to try and meet this week, just to touch base and see how everything is going. I'm free: Anytime Tue., Wednesday after 4pm, Thursday 1-4pm</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>I can do Tuesday 11-2:30, 4+, Wednesday 5-6, Thursday 11-2:30</td>
</tr>
<tr>
<td>P2</td>
<td>I can do anytime Tuesday (just before or after lunch is best), not Wednesday, or Thursday afternoon.</td>
</tr>
<tr>
<td>S2</td>
<td>I'm free Tuesday and Thursday from 2:45-4 or so, and also Wednesday any time after 3.</td>
</tr>
<tr>
<td>S3</td>
<td>I can meet from 4-5 on Tuesday or Wednesday after 5.</td>
</tr>
</tbody>
</table>
### Applications

#### Scheduling

- manufacturing
- Hubble telescope time usage
- Airlines
- Cryptography
- computer vision (image interpretation)
- ...

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### Why CSPs?

“**Constraint programming represents one of the closest approaches computer science has yet made to the Holy Grail of programming: the user states the problem, the computer solves it.**”

Eugene C. Freuder, Constraints, April 1997

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### Why CSPs?

If you can represent it in this standard way (set of variables with a domain of values and constraints), the successor function and goal test can be written in a generic way that applies to **all** CSPs.

We can develop effective generic heuristics that require **no domain specific expertise**.

The **structure** of the constraints can be used to simplify the solution process.
Defining CSP problems

1. variables
2. domains of the variables
3. constraints

Example: 8-Queens Problem

• 8 variables $x_1, x_2, \ldots, x_8$
• Domain for each variable: $\{1, 2, \ldots, 8\}$
• Constraints are of the forms:
  – row constraints: $x_i \neq x_j$ for all $i, j$ where $j \neq i$
  – diagonal constraints: $|x_i - x_j|$, for all $i, j$ where $j \neq i$

Example: Map Coloring

• 7 variables $\{WA, NT, SA, Q, NSW, V, T\}$
• Each variable has the same domain $\{\text{red, green, blue}\}$
• No two adjacent variables have the same value:
  $WA = NT$, $WA = SA$, $NT = SA$, $SA = Q$, $SA = NSW$, $SA = Q = NSW$, $NSW = V$

CSP Example: Cryptarithmic puzzle

Variables: $F, T, U, W, R, O, X_1, X_2, X_3$
Domains: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
Constraints
  $\forall i \neq j (F, T, U, W, R, O)$$
  O + O = R + 10 \cdot X_1$, etc.
**Example: Task Scheduling**

T1 must be done during T3
T2 must be achieved before T1 starts
T2 must overlap with T3
T4 must start after T1 is complete

**Many different constraint types**

Unary constraints: involve only a single variable (x, in green)

Binary constraints: involve two variables

Higher order constraints: involve 3 or more variables (e.g. `all-diff(a,b,c,d,e)`)
  - all higher order constraints can be rewritten as binary constraints by introducing additional variables!

Preference constraints - no absolute - they indicate which solutions are preferred
  - I can meet between 3-4, but I’d prefer to meet between 2-3
  - Electricity is cheaper at night
  - Workers prefer to work in the daytime

**Constraint Graph**

Binary constraints

Two variables are adjacent or neighbors if they are connected by an edge or an arc

**CSP as a Search Problem**

Initial state:
  - {} no assignments

Successor function:
  - any assignment to an unassigned variable that does not conflict

Goal test:
  - all variables assigned to?

Max search depth?
  - number of variables
CSP as search

CSP as search

CSP as search

CSP as search
CSP as search

Backtracking Algorithm

CSP-BACKTRACKING(PartialAssignment \( a \))
- If \( a \) is complete
  - return \( a \)
- \( x \) \( \leftarrow \) select an unassigned variable
- \( D \) \( \leftarrow \) select an ordering for the domain of \( x \)
- For each value \( v \) in \( D \) do
  - If \( v \) is consistent with \( a \) then
    - Add \((x = v)\) to \( a \)
    - result \( \leftarrow \) CSP-BACKTRACKING(\( a \))
  - If result \( \neq \) failure then return result
- Return failure

CSP-BACKTRACKING(\( \{\} \))

Questions

- CSP-BACKTRACKING(PartialAssignment \( a \))
  - If \( a \) is complete
    - return \( a \)
  - \( x \) \( \leftarrow \) select an unassigned variable
  - \( D \) \( \leftarrow \) select an ordering for the domain of \( x \)
  - For each value \( v \) in \( D \) do
    - If \( v \) is consistent with \( a \) then
      - Add \((x = v)\) to \( a \)
      - result \( \leftarrow \) CSP-BACKTRACKING(\( a \))
    - If result \( \neq \) failure then return result
  - Return failure

- Which variable \( x \) should be assigned a value next?
- In which order should its domain \( D \) be sorted?
- How do choices made affect assignments for unassigned variables?

Choice of Variable

- \( x \) \( \leftarrow \) select an unassigned variable

Which variable should we pick?
The most constrained variable, i.e. the one with the fewest remaining values – column 3
Which variable should we start with?

The variable involved with the most constraints - SA

Which value should we pick for Q?

Least constraining value - RED

Least constraining value

Prefer the value that leaves the largest subset of legal values for other unassigned variables

Why CSPs?

Notice that our heuristics work for any CSP problem formulation

- unlike our previous search problems!
- does not require any domain knowledge
  * mancala heuristics
  * straight-line distance
Eliminating wasted search

One of the other important characteristics of CSPs is that we can prune the domain values without actually searching (searching implies guessing).

Our goal is to avoid searching branches that will ultimately dead-end.

How can we use the information available at early on to help with this process?

Constraint Propagation …

… is the process of determining how the possible values of one variable affect the possible values (domains) of other variables.

Forward Checking

After a variable $X$ is assigned a value $v$, look at each unassigned variable $Y$ that is connected to $X$ by a constraint and delete from $Y$’s domain any value that is inconsistent with $v$.

Forward checking

Can we detect inevitable failure early?

– And avoid it later?

Forward checking idea: keep track of remaining legal values for unassigned variables.

Terminate search when any variable has no legal values.
Pick red for WA… how does it change the domains?

Pick green for Q… how does it change the domains?
Pick blue for V... how does it change the domains?

Only picked 3 colors, but already know we're at a dead end!

After just selecting 2... anything wrong with this?
Removal of Arc Inconsistencies

Given two variables $x_j$ and $x_k$ that are connected by some constraint.

We have the current remaining domains $D_{x_j}$ and $D_{x_k}$.

For every possible label in $D_{x_j}$:
- if using that label leaves no possible labels in $D_{x_k}$
  - Then get rid of that possible label.

See full pseudocode in the book.

Arc consistency: AC-3 algorithm

What happens if we remove a possible value during an arc consistency check?
- may cause other domains to change!

When do we stop?
- keep running repeatedly until no inconsistencies remain
- can get very complicated to keep track of which to check.

Arc consistency: AC-3 algorithm

systematic way to keep track of which arcs still need to be checked.

AC-3
- keep track of the set of possible constraints/arcs that may need to be checked
- grab one from this set
- if we make changes to variable’s domain, add all of it’s constraints into the set
- keep doing this until no constraints exist

Solving a CSP

Search:
- can find good solutions, but must examine non-solutions along the way

Constraint Propagation:
- can rule out non-solutions, but this is not the same as finding solutions

Interweave constraint propagation and search
- Perform constraint propagation at each search step.
What can we remove with forward checking?

Anything else with arc consistency?

Anything else?

Anything else?

4-Queens Problem

X1 \{1,2,3,4\}

X2 \{1,2,3,4\}

X3 \{1,2,3,4\}

X4 \{1,2,3,4\}

X1 \{1,2,3,4\}

X2 \{1,2,3,4\}

X3 \{1,2,3,4\}

X4 \{1,2,3,4\}

X1 \{1,2,3,4\}

X2 \{1,2,3,4\}

X3 \{1,2,3,4\}

X4 \{1,2,3,4\}

X1 \{1,2,3,4\}

X2 \{1,2,3,4\}

X3 \{1,2,3,4\}

X4 \{1,2,3,4\}

Can't have X2 = 3!

Can't have X3 = 4!

Can't have X3 = 2!
4-Queens Problem

Technically no search over values was involved. Only looked at constraints.

- Can't have $X_3 = 3$
- Can't have $X_4 = 1$ or $X_4 = 4$
4-Queens Problem

X1 \{2,3,4\}
X2 \{4\}
X3 \{1\}
X4 \{3\}

4-Queens Problem

X1 \{2,3,4\}
X2 \{4\}
X3 \{1\}
X4 \{3\}

Only searched 2 nodes!

CSP Summary

Key: allow us to use heuristics that are problem independent

CSP as a search problem
- Backtracking algorithm
- General heuristics

Forward checking

Constraint propagation

Interweaving CP and backtracking

Edge Labeling in Computer Vision

How do you know what the 3-D shape looks like?
Information about the other edges constrains the possibilities
Labels of Edges

Convex edge:
- two surfaces intersecting at an angle greater than 180°
- often, “sticking out”, “towards us”

Concave edge
- two surfaces intersecting at an angle less than 180°
- often, “folded in”, “away from us”

+ convex edge, both surfaces visible

− concave edge, both surfaces visible

← convex edge, only one surface is visible and it is on the right side of ←

Edge Labeling

Junction Label Sets

(Waltz, 1975; Mackworth, 1977)
Edge Labeling as a CSP

A variable is associated with each junction

The domain of a variable is the label set of the corresponding junction

Each constraint imposes that the values given to two adjacent junctions give the same label to the joining edge