Recurrences

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cs302
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Administrative

- Assignment 1
  - Proof should tell a clear story
  - Proof by induction: follow the steps outlined in class (see the notes)
- Assignment 2?
- Assignment 3 out today (start early!)
- Latex?
- My view on homework…

Divide and Conquer

Divide: Break the problem into smaller sub-problems

Conquer: Solve the sub-problems. Generally, this involves waiting for the problem to be small enough that it is trivial to solve (i.e. 1 or 2 items)

Combine: Given the results of the solved sub-problems, combine them to generate a solution for the complete problem

Divide and Conquer: some thoughts

Often, the sub-problem is the same as the original problem

Dividing the problem in half frequently does the job

May have to get creative about how the data is split

Splitting tends to generate run times with log n in them

Divide and conquer

One approach:

- Pretend like you have a working version of your function, but it only works on smaller sub-problems
- If you split up the current problem in some way (e.g. in half) and solved those sub-problems, how could you then get the solution to the larger problem?

Divide and Conquer: Sorting

How should we split the data?

What are the sub-problems we need to solve?

How do we combine the results from these sub-problems?

MergeSort

```
MERGE-SORT(A)
if length[A] == 1
        return A
else
        q ← ⌊length[A] /2⌋
        create arrays L[1..q] and R[q + 1.. length[A]]
        copy A[1..q] to L
        copy A[q + 1.. length[A]] to R
        LS ← MERGE-SORT(L)
        RS ← MERGE-SORT(R)
        return MERGE(LS, RS)
```

MergeSort: Merge

Assuming L and R are sorted already, merge the two to create a single sorted array

```
MERGE(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
        if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
                B[k] ← L[i]
                i ← i + 1
        else
                B[k] ← R[j]
                j ← j + 1
return B
```

Merge

L: 1 3 5 8 R: 2 4 6 7

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Merge

L: 1 3 5 8 R: 2 4 6 7

B:

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Merge

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Merge

L: 1 3 5 8 R: 2 4 6 7

B:

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Merge

L: 1 3 5 8   R: 2 4 6 7

B: 1

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Merge

L: 1 3 5 8   R: 2 4 6 7

B: 1

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Merge

L: 1 3 5 8   R: 2 4 6 7

B: 1 2

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Merge

L: 1 3 5 8 R: 2 4 6 7

B: 1 2 3

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B: 1 2 3 4

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B: 1 2 3 4 5

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B: 1 2 3 4 5 6

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Merge

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Merge

L: 1 3 5 8 R: 2 4 6 7

B: 1 2 3 4 5 6 7 8

```
MERGE(L, R)
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```

Merge

Does the algorithm terminate?

```
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create array B of length length[L] + length[R]
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for k ← 1 to length[B]
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return B
```

Merge

Is it correct?

Loop invariant:

```
MERGE(L, R)
1  create array B of length length[L] + length[R]
2  i ← 1
3  j ← 1
4  for k ← 1 to length[B]
5          if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
6                  B[k] ← L[i]
7                  i ← i + 1
8          else
9                  B[k] ← R[j]
10                 j ← j + 1
11 return B
```

Merge

Is it correct?

Loop invariant: At the beginning of the **for** loop of lines 4-10 the first k-1 elements of B are the smallest *k-1* elements from L and R in sorted order.


```
MERGE(L, R)
1  create array B of length length[L] + length[R]
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4  for k ← 1 to length[B]
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Merge

Running time?


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Merge

Running time? $\Theta(n)$ - linear


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MergeSort

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if length[A] == 1
        return A
else
        q ← ⌊length[A] /2⌋
        create arrays L[1..q] and R[q + 1.. length[A]]
        copy A[1..q] to L
        copy A[q + 1.. length[A]] to R
        LS ← MERGE-SORT(L)
        RS ← MERGE-SORT(R)
        return MERGE(LS, RS)
```

Merge-Sort

Running time?

$$T(n) = \begin{cases} c & \text{if } n \text{ is small} \\ 2T(n/2) + D(n) + C(n) & \text{otherwise} \end{cases}$$

D(n): cost of splitting (dividing) the data

C(n): cost of merging/combining the data

Merge-Sort

Running time?

$$T(n) = \begin{cases} c & \text{if } n \text{ is small} \\ 2T(n/2) + D(n) + C(n) & \text{otherwise} \end{cases}$$

D(n): cost of splitting (dividing) the data - linear Θ(n)

C(n): cost of merging/combining the data – linear Θ(n)

Merge-Sort

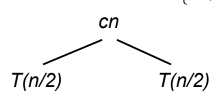
Running time?

$$T(n) = \begin{cases} c & \text{if } n \text{ is small} \\ 2T(n/2) + cn & \text{otherwise} \end{cases}$$

Which is?

Merge-Sort

$$T(n)=\begin{cases} c & \text{if } n \text{ is small} \\ 2T(n/2)+cn & \text{otherwise} \end{cases}$$



Merge-Sort

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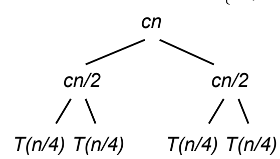

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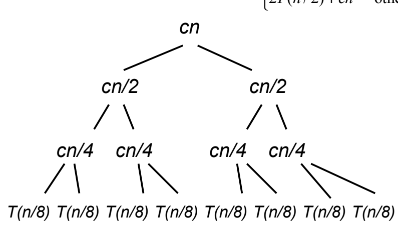

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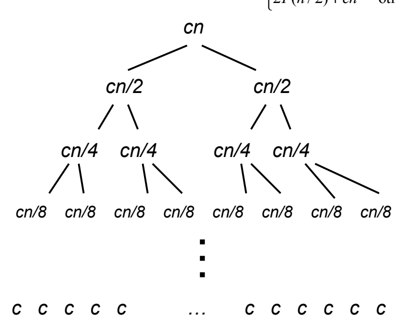
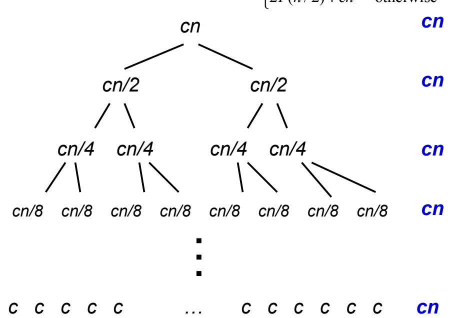


Merge-Sort

$$T(n) = \begin{cases} c & \text{if } n \text{ is small} \\ 2T(n/2) + cn & \text{otherwise} \end{cases}$$



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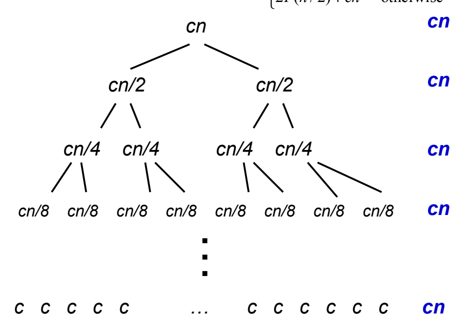


Depth?

Merge-Sort

We can calculate the depth, by determining when the recursion gets to down to a small problem size, e.g. 1

At each level, we divide by 2

$$\frac{n}{2^d} = 1$$

$$2^d = n$$

$$\log 2^d = \log n$$

$$d \log 2 = \log n$$

$$d = \log_2 n$$

Merge-Sort

$$T(n) = \begin{cases} c & \text{if } n \text{ is small} \\ 2T(n/2) + cn & \text{otherwise} \end{cases}$$

Running time?

- Each level costs *cn*
- log *n* levels

cn log *n* = Θ(*n* log *n*)

Recurrence

A function that is defined with respect to itself on smaller inputs

$$T(n) = 2T(n/2) + n$$

$$T(n) = 16T(n/4) + n$$

$$T(n) = 2T(n-1) + n^2$$

Why are we interested in recurrences?

Computational cost of divide and conquer algorithms

$$T(n) = aT(n/b) + D(n) + C(n)$$

- *a* subproblems of size *n/b*
- *D(n)* the cost of dividing the data
- *C(n)* the cost of recombining the subproblem solutions

In general, the runtimes of most recursive algorithms can be expressed as recurrences

The challenge

Recurrences are often easy to define because they mimic the structure of the program

But… they do not directly express the computational cost, i.e. n, n^2, …

We want to remove self-recurrence and find a more understandable form for the function

Three approaches

Substitution method: when you have a good guess of the solution, prove that it's correct

Recursion-tree method: If you don't have a good guess, the recursion tree can help. Then solve with substitution method.

Master method: Provides solutions for recurrences of the form:

$$T(n) = aT(n/b) + f(n)$$

Substitution method

Guess the form of the solution
Then prove it's correct by induction

$$T(n) = T(n/2) + d$$

Halves the input then constant amount of work

Guesses?

Substitution method

Guess the form of the solution
Then prove it's correct by induction

$$T(n) = T(n/2) + d$$

Halves the input then constant amount of work
Similar to binary search:

Guess: O($\log_2$ n)

Proof?

$$T(n) = T(n/2) + d = O(\log_2 n)?$$

Ideas?

Proof?

$$T(n) = T(n/2) + d = O(\log_2 n)?$$

Proof by induction!
- Assume it's true for smaller T(k), i.e. k < n
- prove that it's then true for current T(n)

$$T(n) = T(n/2) + d$$

Assume *T(k) = O($\log_2 k$)* for all *k < n*
Show that *T(n) = O($\log_2 n$)*

From our assumption, *T(n/2) = O($\log_2 n$):*

$$O(g(n)) = \left\{ f(n): \begin{array}{l} \text{there exists positive constants } c \text{ and } n \text{ such that} \\ 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \end{array} \right\}$$

From the definition of big-*O*: *T(n/2) ≤ c $\log_2(n/2)$*

How do we now prove T(n) = O(log n)?

$$T(n) = T(n/2) + d$$

To prove that *T(n) = O($\log_2 n$)* identify the appropriate constants:

$$O(g(n)) = \left\{ f(n): \begin{array}{l} \text{there exists positive constants } c \text{ and } n \text{ such that} \\ 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \end{array} \right\}$$

i.e. some constant *c* such that *T(n) ≤ c $\log_2 n$*

$$\begin{aligned} T(n) &= T(n/2) + d \\ &\le c\log_2(n/2) + d && \text{from our inductive hypothesis} \\ &\le c\log_2 n - c\log_2 2 + d \\ &\le c\log_2 n - c + d && \text{residual} \\ &\le c\log_2 n \end{aligned}$$

if *c ≥ d*

Base case?

For an inductive proof we need to show two things:

- Assuming it's true for *k < n* show it's true for *n*
- *Show that it holds for some base case*

What is the base case in our situation?

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \text{ is small} \\ T(n/2) + d & \text{otherwise} \end{cases}$$

$$T(n) = T(n-1) + n$$

Guess the solution?

- At each iteration, does a linear amount of work (i.e. iterate over the data) and reduces the size by one at each step
- *O(n^2)*

Assume *T(k) = O(k^2)* for all *k < n*

- again, this implies that *T(n-1) ≤ c$(n-1)^2$*

Show that *T(n) = O(n^2)*, i.e. *T(n) ≤ cn^2*

$$
\begin{aligned}
T(n) &= T(n-1) + n \\
&\le c(n-1)^2 + n \quad \text{from our inductive hypothesis} \\
&= c(n^2 - 2n + 1) + n \\
&= cn^2 - 2cn + c + n \quad \text{residual} \\
&\le cn^2
\end{aligned}
$$

if

$$
\begin{aligned}
-2cn + c + n &\le 0 \\
-2cn + c &\le -n \\
c(-2n+1) &\le -n \\
c &\ge \frac{n}{2n-1} \\
c &\ge \frac{1}{2-1/n}
\end{aligned}
$$

which holds for any $c \ge 1$ for $n \ge 1$

$$T(n) = 2T(n/2) + n$$

Guess the solution?

- Recurses into 2 sub-problems that are half the size and performs some operation on all the elements
- $O(n \log n)$

What if we guess wrong, e.g. $O(n^2)$?

Assume $T(k) = O(k^2)$ for all $k < n$

- again, this implies that $T(n/2) \le c(n/2)^2$

Show that $T(n) = O(n^2)$

$$
\begin{aligned}
T(n) &= 2T(n/2) + n \\
&\le 2c(n/2)^2 + n \quad \text{from our inductive hypothesis} \\
&= 2cn^2/4 + n \\
&= 1/2cn^2 + n \\
&= cn^2 - (1/2cn^2 - n) \quad \text{residual} \\
&\le cn^2
\end{aligned}
$$

if

$$
\begin{aligned}
-(1/2cn^2 - n) &\le 0 \\
-1/2cn^2 + n &\le 0 \\
cn &\ge 2
\end{aligned}
$$

overkill?

$$T(n) = 2T(n/2) + n$$

What if we guess wrong, e.g. $O(n)$?

Assume $T(k) = O(k)$ for all $k < n$

- again, this implies that $T(n/2) \le c(n/2)$

Show that $T(n) = O(n)$

$$
\begin{aligned}
T(n) &= 2T(n/2) + n \\
&\le 2cn/2 + n \\
&= cn + n \\
&\le cn
\end{aligned}
$$

factor of n so we can just roll it in?

$$T(n) = 2T(n/2) + n$$

What if we guess wrong, e.g. O(*n*)?

Assume $T(k) = O(k)$ for all $k < n$

- again, this implies that $T(n/2) \le c(n/2)$

Show that $T(n) = O(n)$

$$T(n) = 2T(n/2) + n$$
$$\le 2cn/2 + n$$
$$= cn + n$$
$$\le cn$$

Must prove the exact form!

$cn+n \le cn$??

~~factor of n so we can just roll it in?~~

$$T(n) = 2T(n/2) + n$$

Prove $T(n) = O(n \log_2 n)$

Assume $T(k) = O(k \log_2 k)$ for all $k < n$

- again, this implies that $T(k) = ck \log_2 k$

Show that $T(n) = O(n \log_2 n)$

$$T(n) = 2T(n/2) + n$$
$$\le 2cn/2 \log(n/2) + n$$
$$\le cn(\log_2 n - \log_2 2) + n$$
$$\le cn \log_2 n - cn + n$$

residual

$$\le cn \log_2 n$$

if $cn \ge n$, $c > 1$

Changing variables

$$T(n) = 2T(\sqrt{n}) + \log n$$

Guesses?

We can do a variable change: let $m = \log_2 n$ (or $n = 2^m$)

$$T(2^m) = 2T(2^{m/2}) + m$$

Now, let $S(m)=T(2^m)$

$$S(m) = 2S(m/2) + m$$

Changing variables

$$S(m) = 2S(m/2) + m$$

Guess? $S(m) = O(m \log m)$

$$T(n) = T(2^m) = S(m) = O(m \log m)$$

substituting m=log n

$$T(n) = O(\log n \log \log n)$$

Recursion Tree

Guessing the answer can be difficult

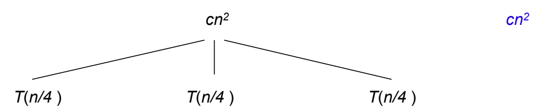
$$T(n) = 3T(n/4) + n^2$$
$$T(n) = T(n/3) + 2T(2n/3) + cn$$

The recursion tree approach

- Draw out the cost of the tree at each level of recursion
- Sum up the cost of the levels of the tree
  - Find the cost of each level with respect to the depth
  - Figure out the depth of the tree
  - Figure out (or bound) the number of leaves
- Verify your answer using the substitution method

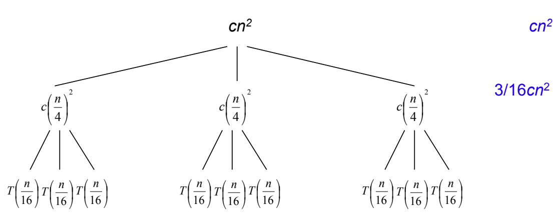
$$T(n) = 3T(n/4) + n^2$$

cost



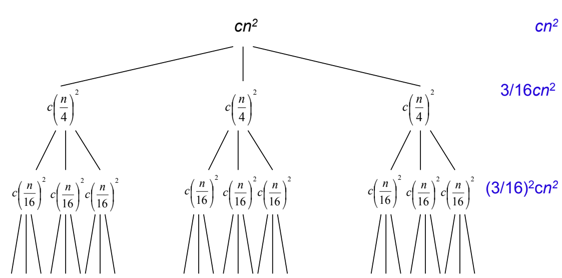
$$T(n) = 3T(n/4) + n^2$$

cost



$$T(n) = 3T(n/4) + n^2$$

cost



What is the cost at each level? $\left(\frac{3}{16}\right)^d cn^2$

What is the depth of the tree?

At each level, the size of the data is divided by 4

$$\frac{n}{4^d} = 1$$

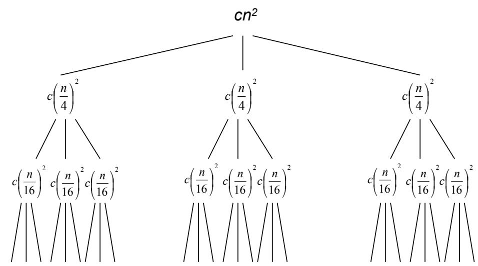
$$\log\left(\frac{n}{4^d}\right) = 0$$

$$\log n - \log 4^d = 0$$

$$d \log 4 = \log n$$

$$d = \log_4 n$$

$$T(n) = 3T(n/4) + n^2$$



$$T(1)$$

How many leaves are there?

How many leaves?

How many leaves are there in a complete ternary tree of depth *d*?

$$3^d = 3^{\log_4 n}$$

Total cost

$$T(n) = cn^2 + \frac{3}{16}cn^2 + \left(\frac{3}{16}\right)^2 cn^2 + \ldots + \left(\frac{3}{16}\right)^{d-1} cn^2 + \Theta(3^{\log_4 n})$$

$$= cn^2 \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{16}\right)^i + \Theta(3^{\log_4 n})$$

$$< cn^2 \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i + \Theta(3^{\log_4 n})$$

$$= \frac{1}{1-(3/16)} cn^2 + \Theta(3^{\log_4 n})$$

$$= \frac{16}{13} cn^2 + \Theta(3^{\log_4 n})$$?

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

let x = 3/16

Total cost $T(n) = \frac{16}{13}cn^2 + \Theta(3^{\log_4 n})$

$$3^{\log_4 n} = 4^{\log_4 3^{\log_4 n}}$$
$$= 4^{\log_4 n \log_4 3}$$
$$= 4^{\log_4 n^{\log_4 3}}$$
$$= n^{\log_4 3}$$

$$T(n) = \frac{16}{13}cn^2 + \Theta(n^{\log_4 3})$$

$$T(n) = O(n^2)$$

Verify solution using substitution

$$T(n) = 3T(n/4) + n^2$$

Assume *T(k) = O(k*²*)* for all *k < n*
Show that *T(n) = O(n*²*)*

Given that *T(n/4) = O((n/4)*²*)*, then

$$O(g(n)) = \left\{ f(n): \begin{array}{l} \text{there exists positive constants } c \text{ and } n \text{ such that} \\ 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \end{array} \right\}$$

$T(n/4) \le c(n/4)^2$

$$T(n) = 3T(n/4) + n^2$$

To prove that Show that *T(n) = O(n*²*)* we need to identify the appropriate constants:

$$O(g(n)) = \left\{ f(n): \begin{array}{l} \text{there exists positive constants } c \text{ and } n \text{ such that} \\ 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \end{array} \right\}$$

i.e. some constant *c* such that $T(n) \le cn^2$

$$T(n) = 3T(n/4) + n^2$$
$$\le 3c(n/4)^2 + n^2$$
$$= cn^2 3/16 + n^2$$
$$\le cn^2$$

if

$$c \ge \frac{16}{13}$$

Master Method

Provides solutions to the recurrences of the form:

$$T(n) = aT(n/b) + f(n)$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$

then $T(n) = \Theta(f(n))$

$$T(n) = 16T(n/4) + n$$

if $f(n) = O(n^{\log_b a-\varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
if $f(n) = \Omega(n^{\log_b a+\varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

a = 16
b = 4
f(n) = *n*

$n^{\log_b a} = n^{\log_4 16}$
$= n^2$

is $n = O(n^{2-\varepsilon})$?
is $n = \Theta(n^2)$?
is $n = \Omega(n^{2+\varepsilon})$?

Case 1: $\Theta(n^2)$

$$T(n) = T(n/2) + 2^n$$

if $f(n) = O(n^{\log_b a-\varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
if $f(n) = \Omega(n^{\log_b a+\varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

a = 1
b = 2
f(n) = 2^n

$n^{\log_b a} = n^{\log_2 1}$
$= n^0$

is $2^n = O(n^{0-\varepsilon})$?
is $2^n = \Theta(n^0)$?
is $2^n = \Omega(n^{0+\varepsilon})$?

Case 3?

is $2^{n/2} \le c2^n$ for $c < 1$?

$$T(n) = T(n/2) + 2^n$$

if $f(n) = O(n^{\log_b a-\varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
if $f(n) = \Omega(n^{\log_b a+\varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

is $2^{n/2} \le c2^n$ for $c < 1$?

Let c = 1/2

$2^{n/2} \le (1/2)2^n$
$2^{n/2} \le 2^{-1}2^n$
$2^{n/2} \le 2^{n-1}$

$T(n) = \Theta(2^n)$

$$T(n) = 2T(n/2) + n$$

if $f(n) = O(n^{\log_b a-\varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
if $f(n) = \Omega(n^{\log_b a+\varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

a = 2
b = 2
f(n) = *n*

$n^{\log_b a} = n^{\log_2 2}$
$= n^1$

is $n = O(n^{1-\varepsilon})$?
is $n = \Theta(n^1)$?
is $n = \Omega(n^{1+\varepsilon})$?

Case 2: $\Theta(n \log n)$

$$T(n) = 16T(n/4) + n!$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$ then $T(n) = \Theta(f(n))$

a = 16
b = 4
f(n) = *n!*

$n^{\log_b a} = n^{\log_4 16}$
$= n^2$

is $n! = O(n^{2-\varepsilon})$?
is $n! = \Theta(n^2)$?
is $n! = \Omega(n^{2+\varepsilon})$?

Case 3?

is $16(n/4)! \le cn!$ for $c < 1$?

$$T(n) = 16T(n/4) + n!$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$ then $T(n) = \Theta(f(n))$

is $16(n/4)! \le cn!$ for $c < 1$?

Let c = 1/2

$cn! = 1/2n!$
$> (n/2)!$

therefore,

$16(n/4)! \le (n/2)! < 1/2n!$

T(*n*) = Θ(*n!*)

$$T(n) = \sqrt{2}T(n/2) + \log n$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$ then $T(n) = \Theta(f(n))$

a = $\sqrt{2}$
b = 2
f(n) = logn

$n^{\log_b a} = n^{\log_2 \sqrt{2}}$
$= n^{\log_2 2^{1/2}}$
$= \sqrt{n}$

is $\log n = O(n^{1/2-\varepsilon})$?
is $\log n = \Theta(n^{1/2})$?
is $\log n = \Omega(n^{1/2+\varepsilon})$?

Case 1: $\Theta(\sqrt{n})$

$$T(n) = 4T(n/2) + n$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$ then $T(n) = \Theta(f(n))$

a = 4
b = 2
f(n) = *n*

$n^{\log_b a} = n^{\log_2 4}$
$= n^2$

is $n = O(n^{2-\varepsilon})$?
is $n = \Theta(n^2)$?
is $n = \Omega(n^{2+\varepsilon})$?

Case 1: $\Theta(n^2)$

Recurrences

$$T(n) = 2T(n/3) + d \qquad T(n) = 7T(n/7) + n$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

$$T(n) = T(n-1) + \log n \qquad T(n) = 8T(n/2) + n^3$$