Graphs

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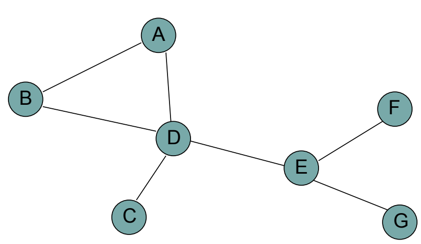
cs302
Spring 2013

Admin

- HW 12 and 13 (and likely 14)
  - You can submit revised solutions to any problem you missed
  - Also submit your original homework
  - I'll give you up to half of the points taken off
  - Because I've given comments/feedback, make sure you explain *why* for simple questions (like run-time)
  - Also, I will expect your answers to be very clear and precise
- HW 15
  - more dynamic programming
  - will involve some programming (you may use any language installed on the lab machines)
  - may work with a partner: you and your partner must always be there when you're working on the assignment

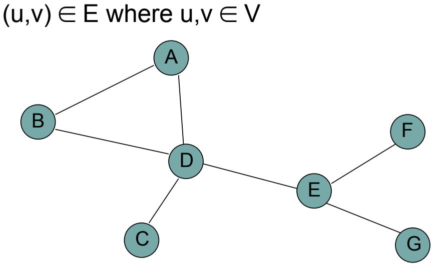
Graphs

What is a graph?



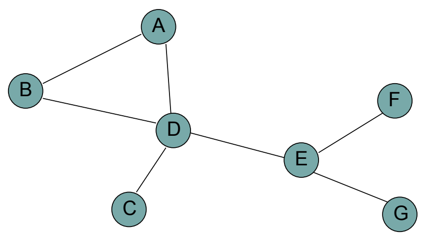
Graphs

A graph is a set of vertices V and a set of edges (u,v) $\in$ E where u,v $\in$ V



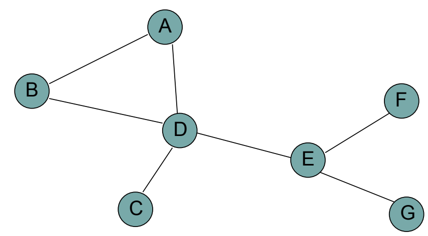
Graphs

How do graphs differ? What are graph characteristics we might care about?



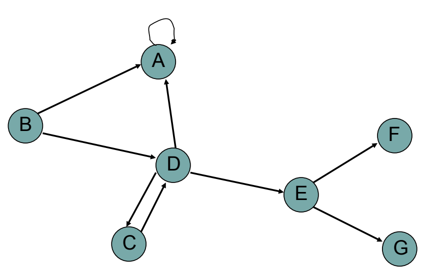
Different types of graphs

Undirected – edges do not have a direction



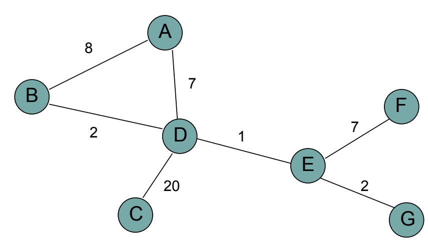
Different types of graphs

Directed – edges **do** have a direction



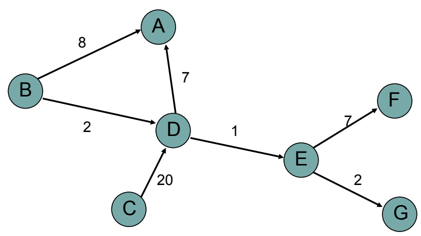
Different types of graphs

Weighted – edges have an associated weight



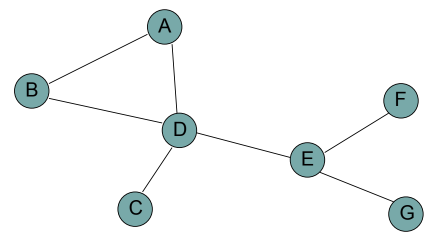
Different types of graphs

Weighted – edges have an associated weight



Terminology

Path – A path is a list of vertices $p_1,p_2,\ldots p_k$ where there exists an edge $(p_i,p_{i+1}) \in E$



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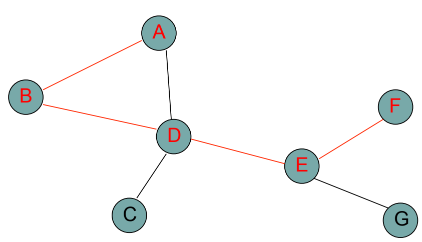
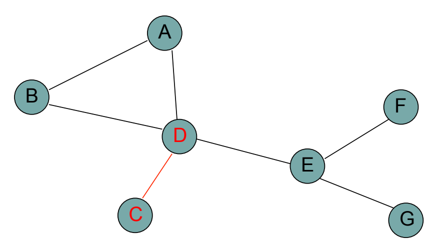
{A, B, D, E, F}



Terminology

Path – A path is a list of vertices $p_1,p_2,\ldots p_k$ where there exists an edge $(p_i,p_{i+1}) \in E$

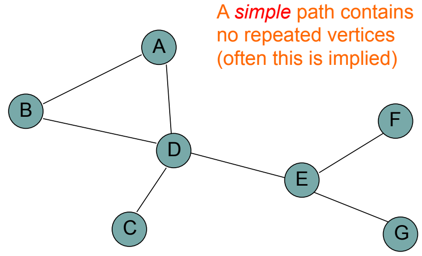
{C, D}



Terminology

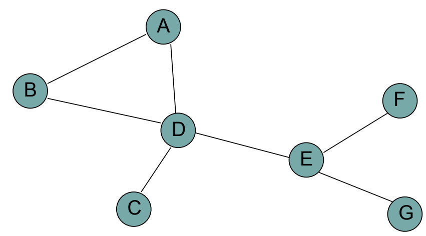
Path – A path is a list of vertices $p_1, p_2, \ldots p_k$ where there exists an edge $(p_i, p_{i+1}) \in E$

A *simple* path contains no repeated vertices (often this is implied)



Terminology

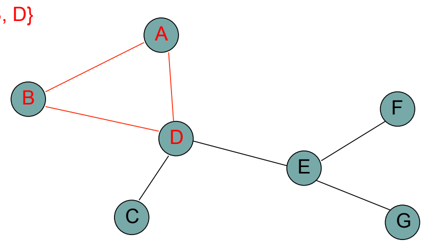
Cycle – A subset of the edges that form a path such that the first and last node are the same



Terminology

Cycle – A subset of the edges that form a path such that the first and last node are the same

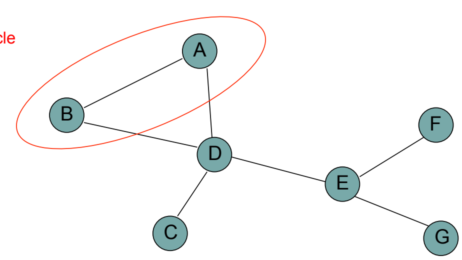
{A, B, D}



Terminology

Cycle – A subset of the edges that form a path such that the first and last node are the same

not a cycle



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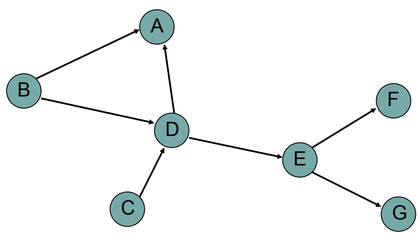

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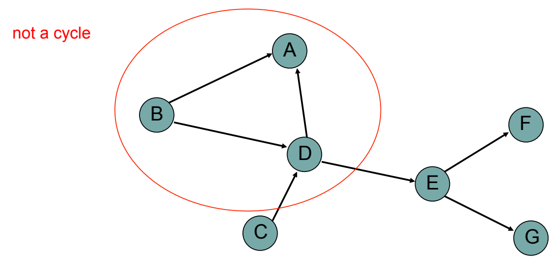
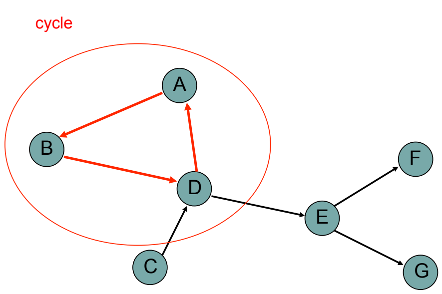


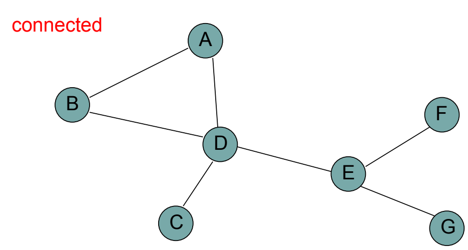
Terminology

Cycle – A path $p_1,p_2,\ldots p_k$ where $p_1 = p_k$



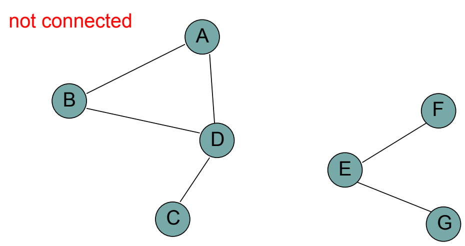
Terminology

Connected – every pair of vertices is connected by a path



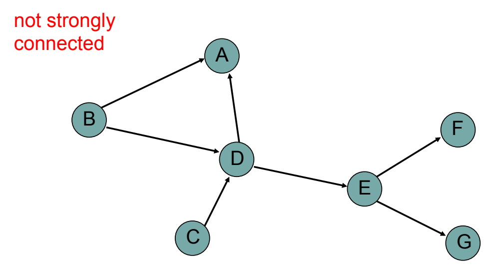
Terminology

Connected (undirected graphs) – every pair of vertices is connected by a path



Terminology

Strongly connected (directed graphs) – Every two vertices are reachable by a path



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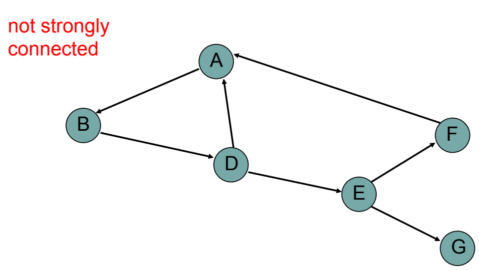
Strongly connected (directed graphs) – Every two vertices are reachable by a path



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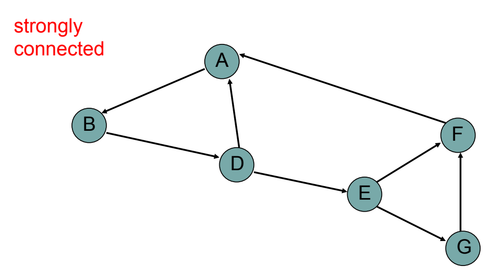
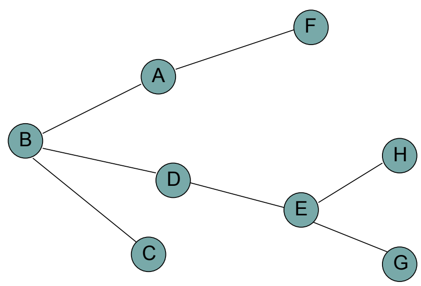


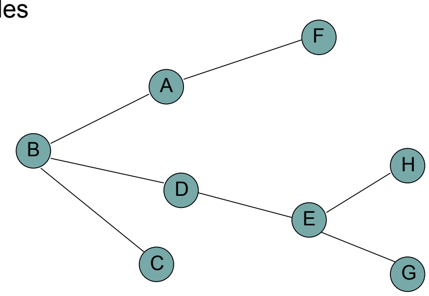
Different types of graphs

What is a tree (in our terminology)?



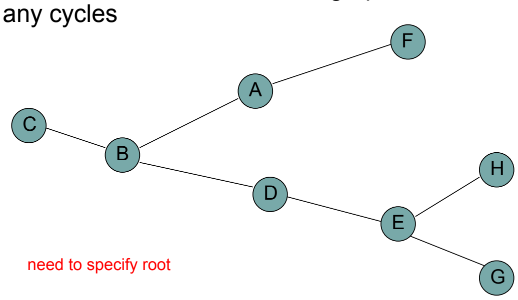
Different types of graphs

Tree – connected, undirected graph without any cycles



Different types of graphs

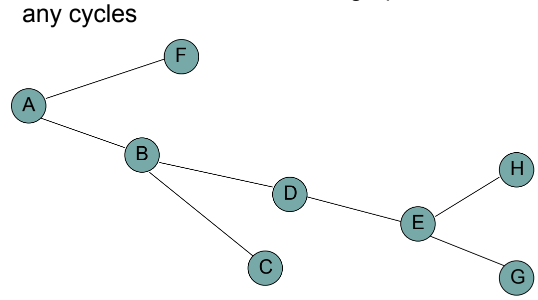
Tree – connected, undirected graph without any cycles



need to specify root

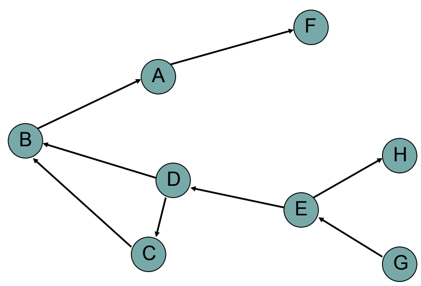
Different types of graphs

Tree – connected, undirected graph without any cycles



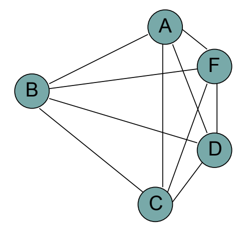
Different types of graphs

DAG – directed, acyclic graph



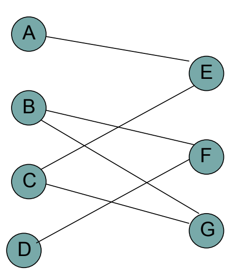
Different types of graphs

Complete graph – an edge exists between every node



Different types of graphs

Bipartite graph – a graph where every vertex can be partitioned into two sets X and Y such that all edges connect a vertex $u \in X$ and a vertex $v \in Y$



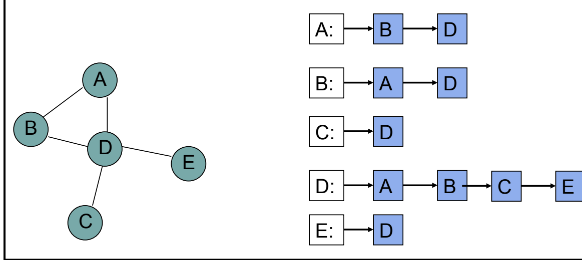
When do we see graphs in real life problems?

- Transportation networks (flights, roads, etc.)
- Communication networks
- Web
- Social networks
- Circuit design
- Bayesian networks

Representing graphs

Representing graphs

Adjacency list – Each vertex u ∈ V contains an adjacency list of the set of vertices v such that there exists an edge (u,v) ∈ E



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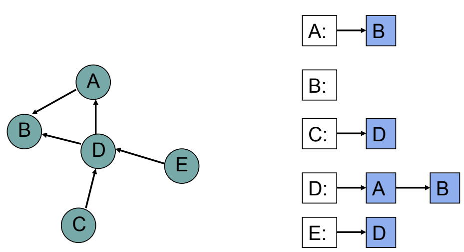
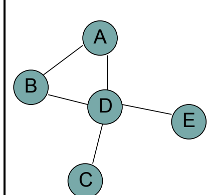


Representing graphs

Adjacency matrix – A |V|x|V| matrix A such that:

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$



	A	**B**	**C**	**D**	**E**
A	0	1	0	1	0
B	1	0	0	1	0
C	0	0	0	1	0
D	1	1	1	0	1
E	0	0	0	1	0

Representing graphs

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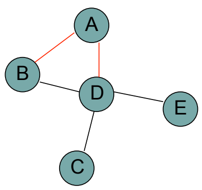

	A	B	C	D	E
A	0	**1**	0	**1**	0
B	1	0	0	1	0
C	0	0	0	1	0
D	1	1	1	0	1
E	0	0	0	1	0

Representing graphs

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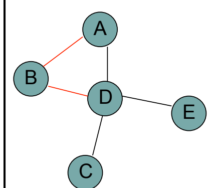

	A	B	C	D	E
A	0	1	0	1	0
B	**1**	0	0	**1**	0
C	0	0	0	1	0
D	1	1	1	0	1
E	0	0	0	1	0

Representing graphs

Adjacency matrix – A |V|x|V| matrix A such that:

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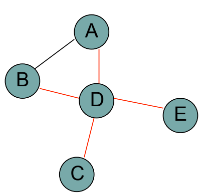

	A	B	C	D	E
A	0	1	0	1	0
B	1	0	0	1	0
C	0	0	0	1	0
D	**1**	**1**	**1**	0	**1**
E	0	0	0	1	0

Representing graphs

Adjacency matrix – A |V|x|V| matrix A such that:

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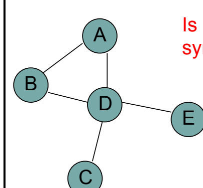


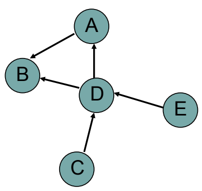
Is it always symmetric?

	A	B	C	D	E
A	0	1	0	1	0
B	1	0	0	1	0
C	0	0	0	1	0
D	1	1	1	0	1
E	0	0	0	1	0

Representing graphs

Adjacency matrix – A |V|x|V| matrix A such that:

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$



	A	B	C	D	E
A	0	1	0	0	0
B	0	0	0	0	0
C	0	0	0	1	0
D	1	1	0	0	0
E	0	0	0	1	0

Adjacency list vs. adjacency matrix

Adjacency list

- Sparse graphs (e.g. web)
- Space efficient
- Must traverse the adjacency list to discover is an edge exists

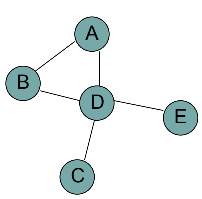
Adjacency matrix

- Dense graphs
- Constant time lookup to discover if an edge exists
- Simple to implement
- For non-weighted graphs, only requires boolean matrix

Can we get the best of both worlds?

Sparse adjacency matrix

Rather than using an adjacency list, use an adjacency hashtable



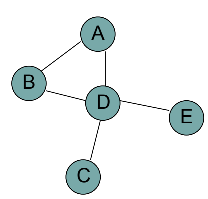
A:	hashtable [B,D]
B:	hashtable [A,D]
C:	hashtable [D]
D:	hashtable [A,B,C,E]
E:	hashtable [D]

Sparse adjacency matrix

Constant time lookup

Space efficient

Not good for dense graphs, why?



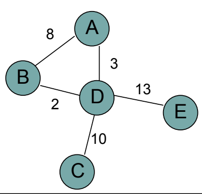
A:	hashtable [B,D]
B:	hashtable [A,D]
C:	hashtable [D]
D:	hashtable [A,B,C,E]
E:	hashtable [D]

Weighted graphs

Adjacency list

- store the weight as an additional field in the list

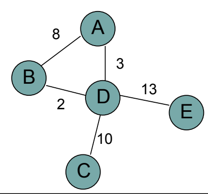
A: → B:8 → D:3



Weighted graphs

Adjacency matrix

$$a_{ij} = \begin{cases} weight & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$



	A	**B**	**C**	**D**	**E**
A	0	8	0	3	0
B	8	0	0	2	0
C	0	0	0	10	0
D	3	2	10	0	13
E	0	0	0	13	0

Graph algorithms/questions

- Graph traversal (BFS, DFS)
- Shortest path from a to b
  - unweighted
  - weighted positive weights
  - negative/positive weights
- Minimum spanning trees
- Are all nodes in the graph connected?
- Is the graph bipartite?
- hw16 and hw17 ☺

Breadth First Search (BFS) on Trees

```
TreeBFS(T)
Enqueue(Q, Root(T))
while !Empty(Q)
        v ← Dequeue(Q)
        Visit(v)
        for all c ∈ Children(v)
                Enqueue(Q, c)
```

Tree BFS

```
TreeBFS(T)
Enqueue(Q, Root(T))
while !Empty(Q)
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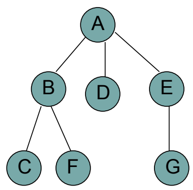

Q:

Tree BFS

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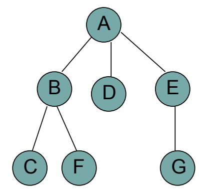

Q: A

Tree BFS

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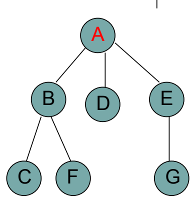

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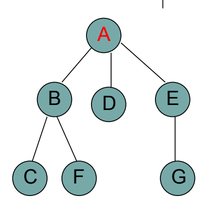
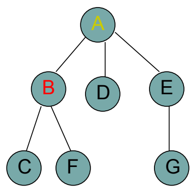


Q: B, D, E

Tree BFS

```
TREEBFS(T)
ENQUEUE(Q, ROOT(T))
while !EMPTY(Q)
    v ← DEQUEUE(Q)
    VISIT(v)
    for all c ∈ CHILDREN(v)
        ENQUEUE(Q, c)
```



Q: D, E

Tree BFS

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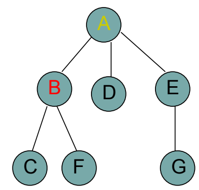

Q: D, E, C, F

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        ENQUEUE(Q, c)
```
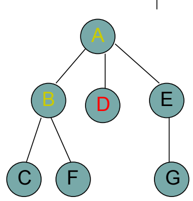
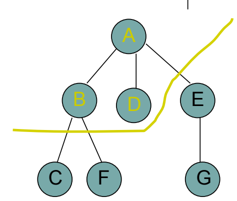



Q: E, C, F

Tree BFS

```
TREEBFS(T)
ENQUEUE(Q, ROOT(T))
while !EMPTY(Q)
    v ← DEQUEUE(Q)
    VISIT(v)
    for all c ∈ CHILDREN(v)
        ENQUEUE(Q, c)
```

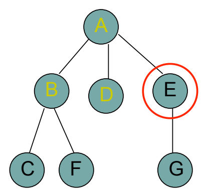


Q: E, C, F

Frontier: the set of vertices that have been visited so far

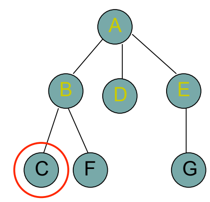
Tree BFS

TreeBFS(T)
Enqueue(Q, Root(T))
while !Empty(Q)
 $v \leftarrow$ Dequeue(Q)
 Visit(v)
 for all $c \in$ Children(v)
  Enqueue(Q, c)



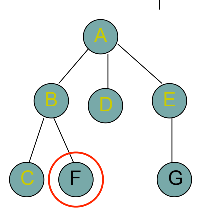
Tree BFS

TreeBFS(T)
Enqueue(Q, Root(T))
while !Empty(Q)
 $v \leftarrow$ Dequeue(Q)
 Visit(v)
 for all $c \in$ Children(v)
  Enqueue(Q, c)



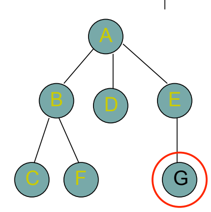
Tree BFS

TreeBFS(T)
Enqueue(Q, Root(T))
while !Empty(Q)
 $v \leftarrow$ Dequeue(Q)
 Visit(v)
 for all $c \in$ Children(v)
  Enqueue(Q, c)



Tree BFS

TreeBFS(T)
Enqueue(Q, Root(T))
while !Empty(Q)
 $v \leftarrow$ Dequeue(Q)
 Visit(v)
 for all $c \in$ Children(v)
  Enqueue(Q, c)



Tree BFS

What order does the algorithm traverse the nodes?

BFS traversal visits the nodes in increasing distance from the root

```
TreeBFS(T)
Enqueue(Q, Root(T))
while !Empty(Q)
        v ← Dequeue(Q)
        Visit(v)
        for all c ∈ Children(v)
                Enqueue(Q, c)
```

Tree BFS

Does it visit all of the nodes?

```
TreeBFS(T)
Enqueue(Q, Root(T))
while !Empty(Q)
        v ← Dequeue(Q)
        Visit(v)
        for all c ∈ Children(v)
                Enqueue(Q, c)
```

Running time of Tree BFS

Adjacency list
- How many times does it visit each vertex?
- How many times is each edge traversed?
- O(|V|+|E|)

Adjacency matrix
- For each vertex visited, how much work is done?
- O(|V|^2)

```
TreeBFS(T)
Enqueue(Q, Root(T))
while !Empty(Q)
        v ← Dequeue(Q)
        Visit(v)
        for all c ∈ Children(v)
                Enqueue(Q, c)
```