Amortized Analysis and Heaps Intro

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cs302
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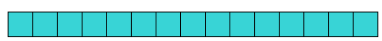
Admin

- Looking for summer researchers in CS at Middlebury
  - Deadline Friday
  - Come talk to me if you want to hear more…
- CS lunch
- Looking ahead…
  - Take-home midterm the week before spring break
    - open notes and book
    - will be timed
  - Review on Tuesday of that week

Admin

- Assignment averages
  - 1: 28.7/30
  - 2: 42.3/49
  - 3: 21.4/23
  - 4: 26.7/32
  - 5: 18.8/20

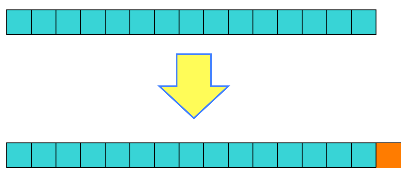
Extensible array



- Sequential locations in memory in linear order
- Elements are accessed via index
  - Access of particular indices is O(1)
- Say we want to implement an array that supports *add* (i.e. *addToBack*)
  - ArrayList or Vector in Java
  - arrays in Python, perl, Ruby, …
- How can we do it?

Extensible array

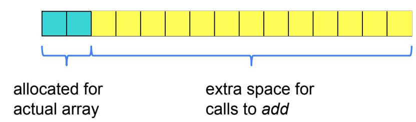
- Idea 1: Each time we call *add*, create a new array one element large, copy the data over and add the element



Running time: Θ(n)

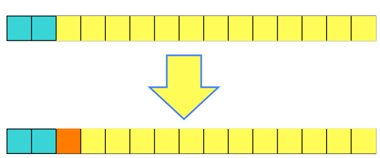
Extensible array

- Idea 2: Allocate extra, unused memory and save room to add elements
- For example: new ArrayList(2)



Extensible array

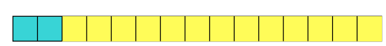
- Idea 2: Allocate extra, unused memory and save room to add elements
- Adding an item:



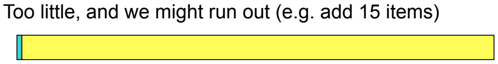
Running time: Θ(1) Problems?

Extensible array

- Idea 2: Allocate extra, unused memory and save room to add elements
- How much extra space do we allocate?



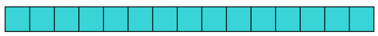
Too little, and we might run out (e.g. add 15 items)



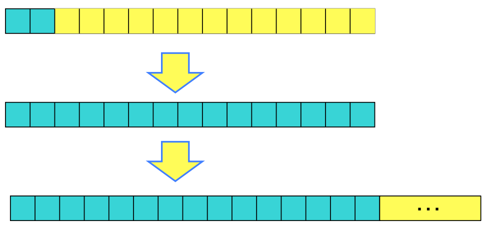
Too much, and we waste lots of memory

Ideas?

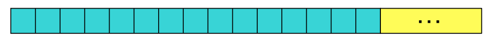
Extensible array



- Idea 3: Allocate some extra memory and when it fills up, allocate some more and copy
- For example: new ArrayList(2)



Extensible array



- Most of the calls to *add* will be O(1)
- What is the ***average*** running time of *add* in the ***worst case***?
  - Note this is different than the average-case running time

Amortized analysis

- There are many situations where the worst case running time is bad
- However, if we average the operations over *n* operations, the average time is more reasonable
- This is called *amortized* analysis
  - This is different than average-case running time, which requires probabilistic reasoning about input
  - The worse case running time doesn't change

Amortized analysis

- Many approaches for calculating the amortized analysis
  - we'll just look at the counting method
  - book has others
- aggregate method
  - figure out the big-O runtime for a sequence of *n* calls
  - divide by *n* to get the average run-time per call

Amortized analysis

What is the aggregate cost of *n* calls?

Let's assume it's O(1) and then prove it

Base case: size 1 array, add an element: O(1)

Inductive case: assume n-1 calls are O(1), show that *n*th call is O(1)

Two cases:
- array need to be doubled
- array does need to be doubled

Amortized analysis

What is the aggregate cost of *n* calls?

- Case 1: doesn't need doubling
  - just add the element into the current array
  - O(1)
- Case 2: need doubling
  - O(n) operation to copy all the data over
  - Overall cost of n-insertion:
    - *n-1*O(1) + O(n) = O(n)*
  - Amortized cost: *O(n)/n = O(1)*

We amortize (spread) the cost of the O(n) operation over all of the previous O(1) operations

Amortized analysis

Another way we could have done the analysis would be to calculate the total cost over *n* operations

$$\text{total\_cost(n)} = \text{basic\_cost(n)} + \text{double\_cost(n)}$$

$$\text{basic\_cost(n)} = O(n) \qquad \text{double\_cost(n)} \le 1+2+4+8+16+\ldots+n = 2n$$

$$\text{total\_cost(n)} = O(n)$$

amortized *O(1)*

Amortized analysis vs. worse case

- What is the worse case of *add*?
  - Still O(n)
  - If you have an application that needs it to be O(1), this implementation **will not work!**
- amortized analysis give you the cost of *n* operations (i.e. average cost) **not** the cost of any individual operation

Extensible arrays

- What if instead of doubling the array, we add instead increase the array by a fixed amount (call it *k*) each time
- Is the amortized run-time still O(1)?
  - No!
  - Why?

Amortized analysis

Consider the cost of *n* insertions for some constant *k*

total_cost(n) = basic_cost(n) + double_cost(n)

basic_cost(n) = $O(n)$

double_cost(n) =k+2k+3k+4k+5k+...+n

$$= \sum_{i=1}^{n/k} ki$$

$$= k \sum_{i=1}^{n/k} i$$

$$= k \frac{\frac{n}{k}\left(\frac{n}{k}+1\right)}{2} = O(n^2)$$

Amortized analysis

Consider the cost of *n* insertions for some constant *k*

$$\text{total\_cost(n)} = O(n) + O(n^2)$$
$$= O(n^2)$$

amortized *O(n)!*

Another set data structure

- We want to support fast lookup and insertion (i.e. faster than linear)
- Arrays can easily made to be fast for one or the other
  - fast search: keep list sorted
    - O(n) insert
    - O(log n) search
  - fast insert: extensible array
    - O(1) insert (amortized)
    - O(n) search

Another set data structure

- Idea: store data in a collection of arrays
  - array *i* has size 2^i
  - an array is either full or empty (never partially full)
  - each array is stored in sorted order
  - no relationship between arrays

Another set data structure

- Which arrays are full and empty arrays are based on the number of elements
  - specifically, binary representation of the number of elements
  - 4 items = 100 = A2-full, A1-empty, A0-empty
  - 11 items = 1011 = A3-full, A2-empty, A1-full, A0-full

A_0: [5]
A_1: [4, 8]
A_2: empty
A_3: [2, 6, 9, 12, 13, 16, 20, 25]

- Lookup: binary search through each array
  - Worse case runtime?

Another set data structure

A_0: [5]
A_1: [4, 8]
A_2: empty
A_3: [2, 6, 9, 12, 13, 16, 20, 25]

- Lookup: binary search through each array
- Worse case: all arrays are full
  - number of arrays = number of digits = log n
  - binary search cost for each array = O(log n)
  - O(log n log n)

Another set data structure

- Insert
  - starting at i = 0
  - current = [item]
  - as long as the level *i* is full
    - merge current with A_i using *merge* procedure
    - store to current
    - A_i = empty
    - i++
  - A_i = current

Insert 5

A_0: empty

- Insert
  - starting at i = 0
  - current = [item]
  - as long as the level *i* is full
    - merge current with A_i using *merge* procedure
    - store to current
    - A_i = empty
    - i++
  - A_i = current

Insert 5

A_0: [5]

- Insert
  - starting at i = 0
  - current = [item]
  - as long as the level *i* is full
    - merge current with A_i using *merge* procedure
    - store to current
    - A_i = empty
    - i++
  - A_i = current

Insert 6

A_0: [5]

- Insert
  - starting at i = 0
  - current = [item]
  - as long as the level *i* is full
    - merge current with A_i using *merge* procedure
    - store to current
    - A_i = empty
    - i++
  - A_i = current

Insert 6

A_0: empty
A_1: [5, 6]

- Insert
  - starting at i = 0
  - current = [item]
  - as long as the level *i* is full
    - merge current with A_i using *merge* procedure
    - store to current
    - A_i = empty
    - i++
  - A_i = current

Insert 12

A_0: empty
A_1: [5, 6]

- Insert
  - starting at i = 0
  - current = [item]
  - as long as the level *i* is full
    - merge current with A_i using *merge* procedure
    - store to current
    - A_i = empty
    - i++
  - A_i = current

Insert 12

A_0: [12]
A_1: [5, 6]

- Insert
  - starting at i = 0
  - current = [item]
  - as long as the level *i* is full
    - merge current with A_i using *merge* procedure
    - store to current
    - A_i = empty
    - i++
  - A_i = current

Insert 4

A_0: [12]
A_1: [5, 6]

- Insert
  - starting at i = 0
  - current = [item]
  - as long as the level *i* is full
    - merge current with A_i using *merge* procedure
    - store to current
    - A_i = empty
    - i++
  - A_i = current

Insert 4

A_0: empty
A_1: empty
A_2: [4, 5, 6, 12]

- Insert
  - starting at i = 0
  - current = [item]
  - as long as the level *i* is full
    - merge current with A_i using *merge* procedure
    - store to current
    - A_i = empty
    - i++
  - A_i = current

Insert 23

A_0: empty
A_1: empty
A_2: [4, 5, 6, 12]

- Insert
  - starting at i = 0
  - current = [item]
  - as long as the level *i* is full
    - merge current with A_i using *merge* procedure
    - store to current
    - A_i = empty
    - i++
  - A_i = current

Insert 23

A_0: [23]
A_1: empty
A_2: [4, 5, 6, 12]

- Insert
  - starting at i = 0
  - current = [item]
  - as long as the level *i* is full
    - merge current with A_i using *merge* procedure
    - store to current
    - A_i = empty
    - i++
  - A_i = current

Another set data structure

- Insert
  - starting at i = 0
  - current = [item]
  - as long as the level *i* is full
    - merge current with A_i using *merge* procedure
    - store to current
    - A_i = empty
    - i++
  - A_i = current

running time?

Another set data structure

- Insert
  - starting at i = 0
  - current = [item]
  - as long as the level *i* is full
    - merge current with A_i using *merge* procedure
    - store to current
    - A_i = empty
    - i++
  - A_i = current

running time?

Insert running time

- Worse case
  - merge at each level
  - 2 + 4 + 8 + … + n/2 + n = O(n)
- There are many insertions that won't fall into this worse case
- What is the amortized worse case for insertion?

insert: amortized analysis

- Consider inserting *n* numbers
  - how many times will A_0 be empty?
  - how many times will we need to merge with A_0?
  - how many times will we need to merge with A_1?
  - how many times will we need to merge with A_2?
  - …
  - how many times will we need to merge with $A_{\log n}$?

insert: amortized analysis

- Consider inserting *n* numbers

	times
how many times will A_0 be empty?	n/2
how many times will we need to merge with A_0?	n/2
how many times will we need to merge with A_1?	n/4
how many times will we need to merge with A_2?	n/8
…	
how many times will we need to merge with $A_{\log n}$?	1

cost of each of these steps?

insert: amortized analysis

- Consider inserting *n* numbers

	times	cost
how many times will A_0 be empty?	n/2	O(1)
how many times will we need to merge with A_0?	n/2	2
how many times will we need to merge with A_1?	n/4	4
how many times will we need to merge with A_2?	n/8	8
…		
how many times will we need to merge with $A_{\log n}$?	1	n

total cost:

insert: amortized analysis

- Consider inserting *n* numbers

	times	cost
how many times will A_0 be empty?	n/2	O(1)
how many times will we need to merge with A_0?	n/2	2
how many times will we need to merge with A_1?	n/4	4
how many times will we need to merge with A_2?	n/8	8
…		
how many times will we need to merge with $A_{\log n}$?	1	n

total cost: log n levels * O(n) each level
O(n log n) cost for *n* inserts
O(log n) amortized cost!

Binary heap

- A binary tree where the value of a parent is greater than or equal to the value of it's children
- Additional restriction: all levels of the tree are **complete** except the last
- Max heap vs. min heap

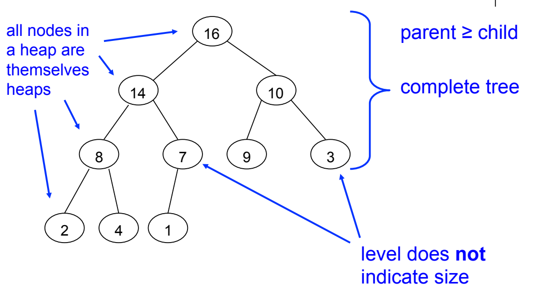
Binary heap - operations

- Maximum(S) - return the largest element in the set
- ExtractMax(S) – Return and remove the largest element in the set
- Insert(S, val) – insert val into the set
- IncreaseElement(S, x, val) – increase the value of element x to val
- BuildHeap(A) – build a heap from an array of elements

Binary heap

How can we represent a heap?

Binary heap - references



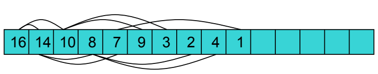
Binary heap - array

PARENT(i)
  return $\lfloor i/2 \rfloor$

LEFT(i)
  return $2i$

RIGHT(i)
  return $2i + 1$

Binary heap - array



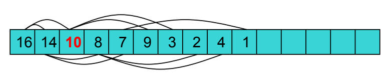
1 2 3 4 5 6 7 8 9 10

PARENT(i)
  return $\lfloor i/2 \rfloor$

LEFT(i)
  return $2i$

RIGHT(i)
  return $2i + 1$

Binary heap - array



1 2 3 4 5 6 7 8 9 10

Left child of A[3]?

PARENT(i)
  return $\lfloor i/2 \rfloor$

LEFT(i)
  return $2i$

RIGHT(i)
  return $2i + 1$

Binary heap - array

16	14	10	8	7	9	3	2	4	1					
1	2	3	4	5	6	7	8	9	10					

PARENT(i)
 return $\lfloor i/2 \rfloor$

LEFT(i)
 return $2i$

RIGHT(i)
 return $2i + 1$

Left child of A[3]?

2*3 = 6

Binary heap - array

16	14	10	8	7	9	3	2	4	1					
1	2	3	4	5	6	7	8	9	10					

PARENT(i)
 return $\lfloor i/2 \rfloor$

LEFT(i)
 return $2i$

RIGHT(i)
 return $2i + 1$

Parent of A[8]?

Binary heap - array

16	14	10	8	7	9	3	2	4	1					
1	2	3	4	5	6	7	8	9	10					

PARENT(i)
 return $\lfloor i/2 \rfloor$

LEFT(i)
 return $2i$

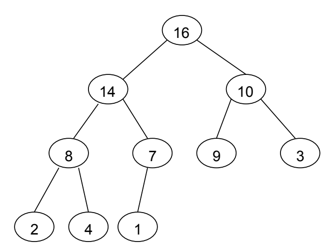
RIGHT(i)
 return $2i + 1$

Parent of A[8]?

$\boxed{\lfloor 8/2 \rfloor = 4}$

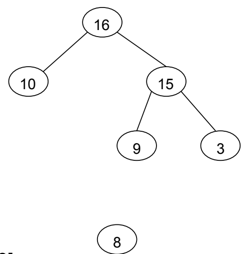
Binary heap - array

16	14	10	8	7	9	3	2	4	1					
1	2	3	4	5	6	7	8	9	10					



Identify the valid heaps

[15, 12, 3, 11, 10, 2, 1, 7, 8]



[20, 18, 10, 17, 16, 15, 9, 14, 13]