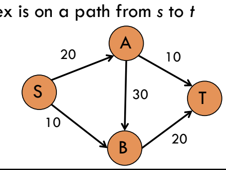
MAX FLOW APPLICATIONS

CS302, Spring 2012 David Kauchak

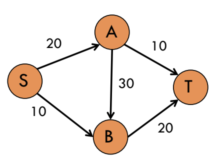
Flow graph/networks

- Flow network
  - directed, weighted graph (V, E)
  - positive edge weights indicating the “capacity” (generally, assume integers)
  - contains a single source $s \in V$ with no incoming edges
  - contains a single sink/target $t \in V$ with no outgoing edges
  - every vertex is on a path from *s* to *t*



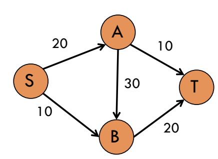
Flow constraints

- in-flow = out-flow for every vertex (except s, t)
- flow along an edge cannot exceed the edge capacity
- flows are positive



Max flow problem

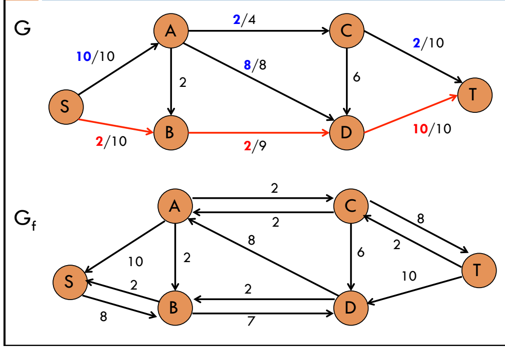
Given a flow network: *what is the maximum flow we can send from s to t that meet the flow constraints?*



The residual graph

- The *residual graph* G_f is constructed from G
- For each edge e in the original graph (G):
  - if *flow(e)* < capacity(e)
    - introduce an edge in G_f with capacity = capacity(e)-flow(e)
    - this represents the remaining flow we can still push
  - if flow(e) > 0
    - introduce an edge in G_f in the *opposite direction* with capacity = flow(e)
    - this represents the flow that we can reroute/reverse

Residual graph



Network flow properties

- If one of these is are true then all are true (i.e. each implies the the others):

- f is a maximum flow
- G_f has no paths from s to t
- |f| = minimum capacity cut

Ford-Fulkerson

```
Ford-Fulkerson(G, s, t)
  flow = 0 for all edges
  Gf = residualGraph(G)
  while a simple path exists from s to t in Gf
    send as much flow along the path as possible
    Gf = residualGraph(G)
  return flow
```

Ford-Fulkerson: runtime?

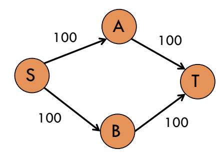
Ford-Fulkerson(G, s, t)
  flow = 0 for all edges
  G_f = residualGraph(G)
  while a simple path exists from s to t in G_f
    send as much flow along path as possible
    G_f = residualGraph(G)
  return flow

Overall runtime? O(max-flow * E)

O(max-flow * E)

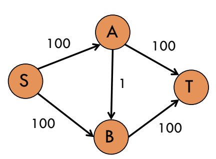
Can you construct a graph that *could get* this running time?

Hint:



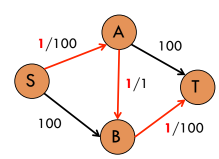
O(max-flow * E)

Can you construct a graph that *could get* this running time?



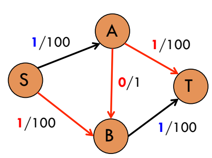
O(max-flow * E)

Can you construct a graph that *could get* this running time?



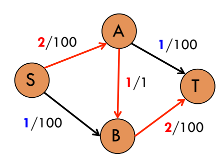
O(max-flow * E)

Can you construct a graph that *could get* this running time?



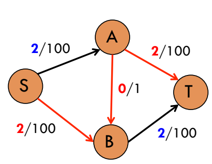
O(max-flow * E)

Can you construct a graph that *could get* this running time?



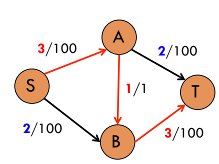
O(max-flow * E)

Can you construct a graph that *could get* this running time?



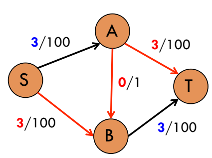
O(max-flow * E)

Can you construct a graph that *could get* this running time?



O(max-flow * E)

Can you construct a graph that *could get* this running time?



What is the problem here?
Could we do better?

Faster variants

- Edmunds-Karp
  - Select the *shortest path* (in number of edges) from s to t in G_f
    - How can we do this?
      - use BFS for search
  - Running time: $O(V E^2)$
    - avoids issues like the one we just saw
    - see the book for the proof
    - or http://www.cs.cornell.edu/courses/CS4820/2011sp/handouts/edmondskarp.pdf
- preflow-push (aka push-relabel) algorithms
  - $O(V^3)$

Other variations…

Method	Complexity
Linear programming	
Ford–Fulkerson algorithm	$O(E \max\| f \|)$
Edmonds–Karp algorithm	$O(VE^2)$
Dinitz blocking flow algorithm	$O(V^2E)$
General push-relabel maximum flow algorithm	$O(V^2E)$
Push-relabel algorithm with *FIFO* vertex selection rule	$O(V^3)$
Dinitz blocking flow algorithm with dynamic trees	$O(VE \log(V))$
Push-relabel algorithm with dynamic trees	$O(VE \log(V^2/E))$
Binary blocking flow algorithm [1]	$O(E \min(V^{2/3}, \sqrt{E}) \log(V^2/E) \log U)$
MPM (Malhotra, Pramodh-Kumar and Maheshwari) algorithm	$O(V^3)$

http://en.wikipedia.org/wiki/Maximum_flow

TABLE I. POLYNOMIAL-TIME ALGORITHMS FOR THE MAXIMUM FLOW PROBLEM[a]

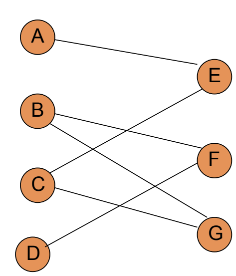
Algorithm no.	Date	Discoverer	Running time	References
1	1969	Edmonds and Karp	$O(nm^2)$	[5]
2	1970	Dinic	$O(n^2m)$	[4]
3	1974	Karzanov	$O(n^3)$	[18]
4	1977	Cherkasky	$O(n^2m^{1/2})$	[3]
5	1978	Malhotra, Pramodh Kumar, and Maheshwari	$O(n^3)$	[21]
6	1978	Galil	$O(n^{5/3}m^{2/3})$	[11]
7	1978	Galil and Naamad; Shiloach	$O(nm(\log n)^2)$	[12, 25]
8	1980	Sleator and Tarjan	$O(nm \log n)$	[27, 28]
9	1982	Shiloach and Vishkin	$O(n^3)$	[26]
10	1983	Gabow	$O(nm \log U)$	[10]
11	1984	Tarjan	$O(n^3)$	[31]
12	1985	Goldberg	$O(n^3)$	[14]
13	1986	Goldberg and Tarjan	$O(nm \log(n^2/m))$	[16, 15]
14	1986	Ahuja and Orlin	$O(nm + n^2 \log U)$	[1]

[a] Algorithm 13 is presented in this paper.

http://akira.ruc.dk/~keld/teaching/algoritmedesign_f03/Artikler/08/Goldberg88.pdf

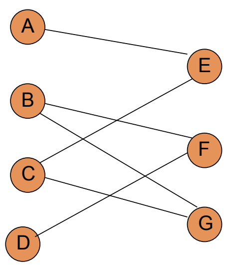
Application: bipartite graph matching

Bipartite graph – a graph where every vertex can be partitioned into two sets X and Y such that all edges connect a vertex $u \in X$ and a vertex $v \in Y$



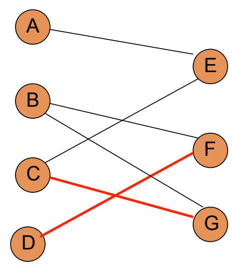
Application: bipartite graph matching

A *matching* M is a subset of edges such that each node occurs **at most once** in M



Application: bipartite graph matching

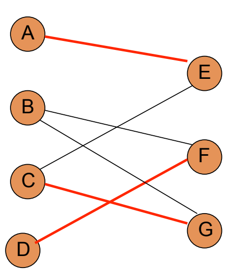
A *matching* M is a subset of edges such that each node occurs **at most once** in M



matching

Application: bipartite graph matching

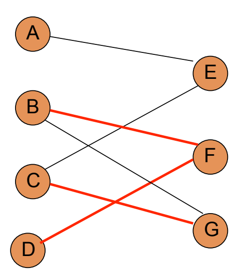
A *matching* M is a subset of edges such that each node occurs **at most once** in M



matching

Application: bipartite graph matching

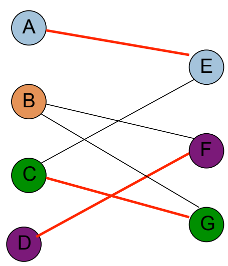
A *matching* M is a subset of edges such that each node occurs **at most once** in M



not a
matching

Application: bipartite graph matching

A *matching* can be thought of as pairing the vertices

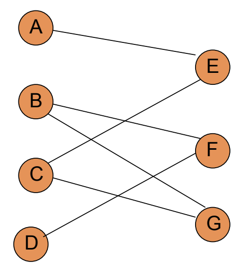


Application: bipartite graph matching

Bipartite matching problem: find the *largest* matching in a bipartite graph

Where might this problem come up?

- CS department has n courses and m faculty
- Every instructor can teach *some* of the courses
- What course should each person teach?
- Anytime we want to match n things with m, but not all things can match

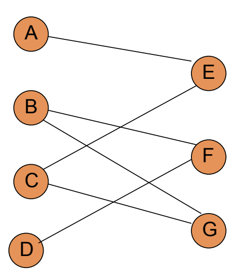


Application: bipartite graph matching

Bipartite matching problem: find the *largest* matching in a bipartite graph

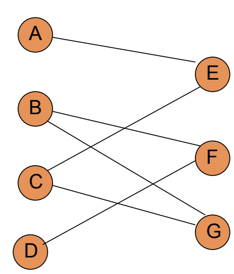
ideas?

- greedy?
- dynamic programming?



Application: bipartite graph matching

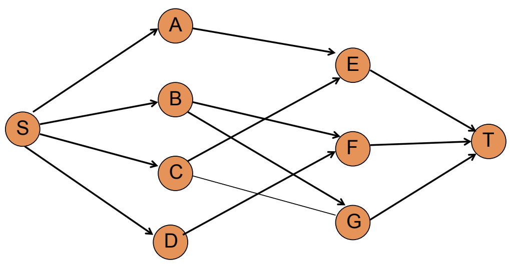
Setup as a flow problem:



Application: bipartite graph matching

Setup as a flow problem:

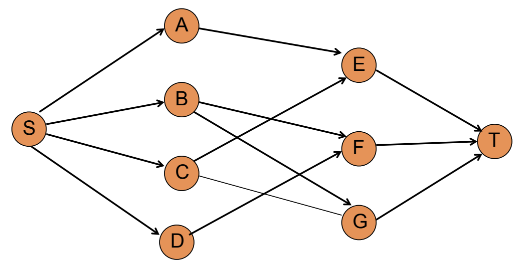
edge weights?



Application: bipartite graph matching

Setup as a flow problem:

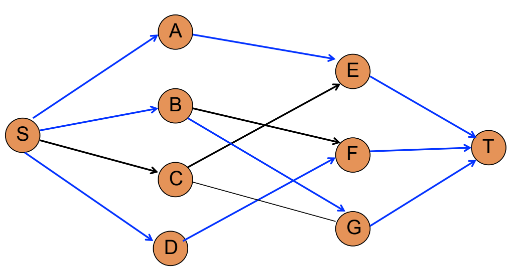
all edge weights are 1



Application: bipartite graph matching

Setup as a flow problem:

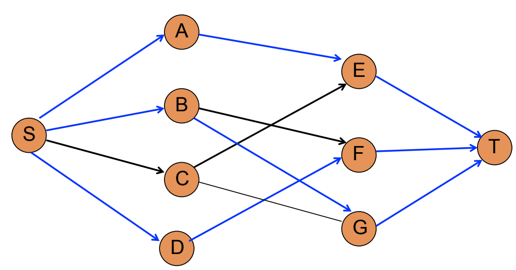
after we find the flow, how do we find the matching?



Application: bipartite graph matching

Setup as a flow problem:

match those nodes with flow between them



Application: bipartite graph matching

- Is it correct?
- Assume it's not
  - there is a better matching
  - because of how we setup the graph flow = # of matches
  - therefore, the better matching would have a higher flow
  - contradiction (max-flow algorithm find maximal!)

Application: bipartite graph matching

- Run-time?
- Cost to build the flow?
  - O(E)
    - each existing edge gets a capacity of 1
    - introduce E new edges (to and from s and t)
- Max-flow calculation?
  - Basic Ford-Fulkerson: O(max-flow * E)
  - Edmunds-Karp: $O(V E^2)$
  - Preflow-push: $O(V^3)$

Application: bipartite graph matching

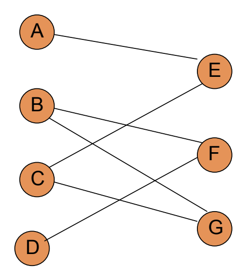
- Run-time?
- Cost to build the flow?
  - O(E)
    - each existing edge gets a capacity of 1
    - introduce E new edges (to and from s and t)
- Max-flow calculation?
  - Basic Ford-Fulkerson: O(max-flow * E)
    - max-flow = O(V)
    - O(V E)

Application: bipartite graph matching

Bipartite matching problem: find the *largest* matching in a bipartite graph

- CS department has n courses and m faculty
- Every instructor can teach *some* of the courses
- What course should each person teach?
- Each faculty can teach at most 3 courses a semester?

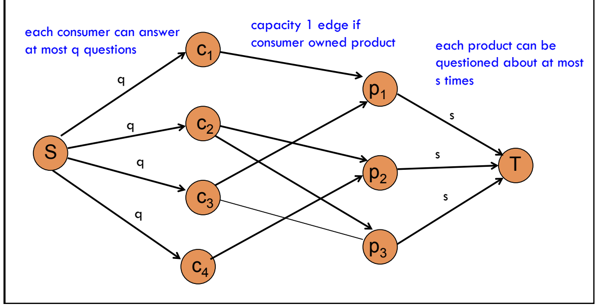
Change the s and t edge weights to 3



Survey Design

- Design a survey with the following requirements:
  - Design survey asking *n* consumers about *m* products
  - Can only survey consumer about a product if they own it
  - Question consumers about at most *q* products
  - Each product should be surveyed at most *s* times
  - Maximize the number of surveys/questions asked
- How can we do this?

Survey Design

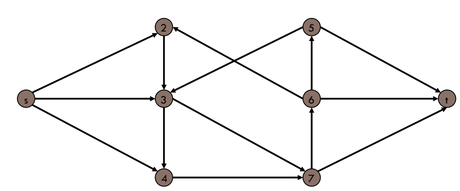


Survey design

- Is it correct?
  - Each of the comments above the flow graph match the problem constraints
  - max-flow finds the maximum matching, given the problem constraints
- What is the run-time?
  - Basic Ford-Fulkerson: O(max-flow * E)
  - Edmunds-Karp: $O(V E^2)$
  - Preflow-push: $O(V^3)$

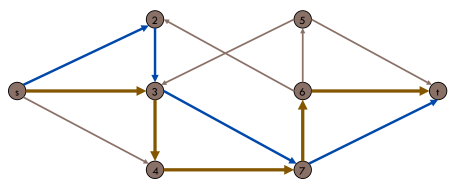
Edge Disjoint Paths

Two paths are edge-disjoint if they have no edge in common



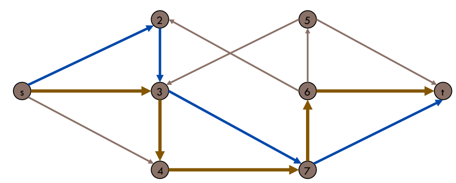
Edge Disjoint Paths

Two paths are edge-disjoint if they have no edge in common



Edge Disjoint Paths Problem

Given a directed graph G = (V, E) and two nodes s and t, find the max number of edge-disjoint paths from s to t



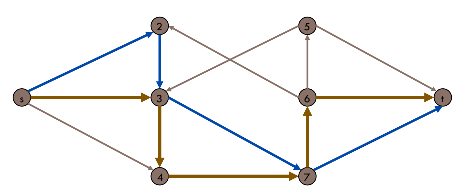
Why might this be useful?

Edge Disjoint Paths Problem

- Given a directed graph G = (V, E) and two nodes s and t, find the max number of edge-disjoint paths from s to t
- Why might this be useful?
  - edges are unique resources (e.g. communications, transportation, etc.)
  - how many *concurrent (non-conflicting)* paths do we have from s to t

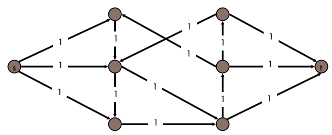
Edge Disjoint Paths

Algorithm ideas?



Edge Disjoint Paths

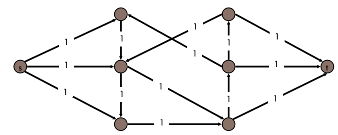
Max flow formulation: assign unit capacity to every edge



What does the max flow represent?
Why?

Edge Disjoint Paths

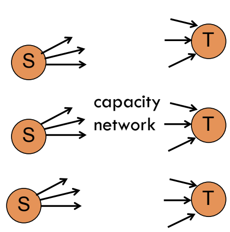
Max flow formulation: assign unit capacity to every edge



- max-flow = maximum number of disjoint paths
- correctness:
  - each edge can have at most flow = 1, so can only be traversed once
  - therefore, each unit out of s represents a separate path to t

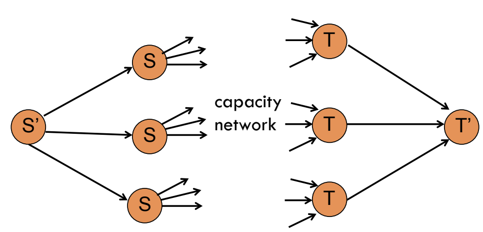
Max-flow variations

What if we have multiple sources and multiple sinks (e.g. the Russian train problem has multiple sinks)?



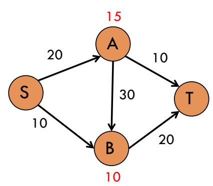
Max-flow variations

Create a new source and sink and connect up with infinite capacities…



Max-flow variations

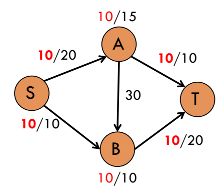
Vertex capacities: in addition to having edge capacities we can also restrict the amount of flow through each vertex



What is the max-flow now?

Max-flow variations

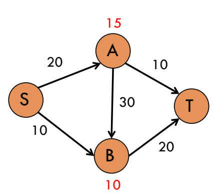
Vertex capacities: in addition to having edge capacities we can also restrict the amount of flow through each vertex



20 units

Max-flow variations

Vertex capacities: in addition to having edge capacities we can also restrict the amount of flow through each vertex

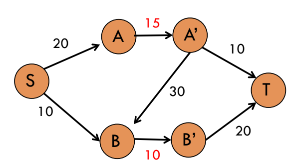


How can we solve this problem?

Max-flow variations

For each vertex v

- create a new node v'
- create an edge with the vertex capacity from v to v'
- move all outgoing edges from v to v'



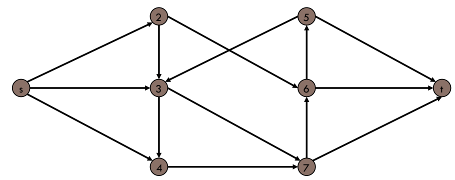
Can you now prove it's correct?

Max-flow variations

- Proof: show that if a solution exists in the modified graph, then a solution exists in the original graph
  - we know that the vertex constraints are satisfied
    - no incoming flow can exceed the vertex capacity since we have a single edge with that capacity from v to v'
  - we can obtain the solution, by collapsing each v and v' back to the original v node
    - in-flow = out-flow since there is only a single edge from v to v'
    - because there is only a single edge from v to v' and all the in edges go in to v and out to v', they can be viewed as a single node in the original graph

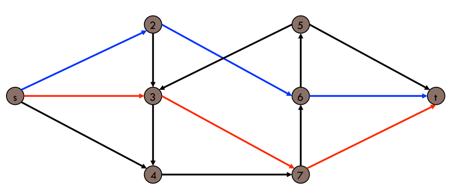
More problems: maximum independent path

Two paths are independent if they have no *vertices* in common



More problems: maximum independent path

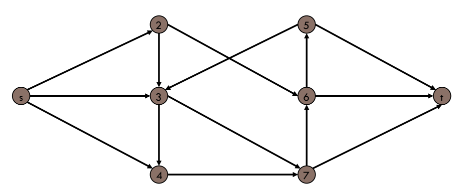
Two paths are independent if they have no *vertices* in common



More problems: maximum independent path

Find the maximum number of independent paths

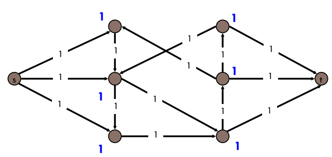
Ideas?



maximum independent path

Max flow formulation:

- assign unit capacity to every edge (though any value would work)
- assign unit capacity to every vertex



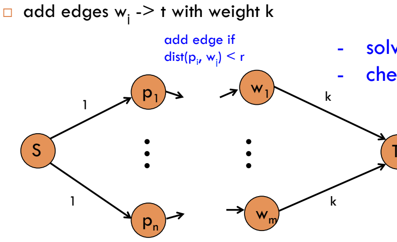
Same idea as the maximum edge-disjoint paths, but now we also constrain the vertices

More problems: wireless network

- The campus has hired you to setup the wireless network
- There are currently *m* wireless stations positioned at various (x,y) coordinates on campus
- The range of each of these stations is *r* (i.e. the signal goes at most distance *r*)
- Any particular wireless station can only host *k* people connected
- You've calculate the *n* most popular locations on campus and have their (x,y) coordinates
- Could the current network support n different people trying to connect at each of the *n* most popular locations (i.e. one person per location)?
- Prove correctness and state run-time

Another matching problem

- n people nodes and n station nodes
- if dist(p_i,w_j) < r then add an edge from pi to wj with weigth 1 (where dist is euclidean distance)
- add edges s -> p_i with weight 1
- add edges w_j -> t with weight k



- solve for max-flow
- check if flow = m

Correctness

- If there is flow from a person node to a wireless node then that person is attached to that wireless node
- if dist(pi,wj) < r then add an edge from pi to wj with weigth 1 (where dist is euclidean distance)
  - only people able to connect to node could have flow
- add edges s -> pi with weight 1
  - each person can only connect to one wireless node
- add edges wj -> t with weight L
  - at most L people can connect to a wireless node
- If flow = m, then every person is connected to a node

Runtime

- $E = O(mn)$: every person is within range of every node
- $V = m + n + 2$
- max-flow $= O(m)$, s has at most m out-flow

- O(max-flow * E) $= O(m^2n)$: Ford-Fulkerson
- $O(VE^2) = O((m+n)m^2n^2)$: Edmunds-Karp
- $O(V^3) = O((m+n)^3)$: preflow-push variant