Big O

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cs302
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Administrative

- Assignment 1: how'd it go?
- Assignment 2: out soon…
- Lab code

Asymptotic notation

- How do you answer the question: "what is the running time of algorithm *x*?"
- We need a way to talk about the computational cost of an algorithm that focuses on the essential parts and ignores irrelevant details
- We' ve seen some of this already:
  - linear
  - $n \log n$
  - n^2

Asymptotic notation

- Precisely calculating the actual steps is tedious and not generally useful
- Different operations take different amounts of time. Even from run to run, things such as caching, etc. cause variations
- Want to identify **categories** of algorithmic runtimes

For example…

- $f_1(n)$ takes n^2 steps
- $f_2(n)$ takes $2n + 100$ steps
- $f_3(n)$ takes $3n+1$ steps

- Which algorithm is better?
- Is the difference between f_2 and f_3 important/ significant?

Runtime examples

	n	$n \log n$	n^2	n^3	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 18 min	10^{25} years
$n = 100$	< 1 sec	< 1 sec	1 sec	1s	10^{17} years	very long
$n = 1000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long
$n = 10,000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long
$n = 100,000$	< 1 sec	2 sec	3 hours	32 years	very long	very long
$n = 1,000,000$	1 sec	20 sec	12 days	31,710 years	very long	very long

(adapted from [2], Table 2.1, pg. 34)

Big O: Upper bound

- $O(g(n))$ is the set of functions:

$$O(g(n)) = \left\{ f(n) : \begin{array}{l} \text{there exists positive constants } c \text{ and } n_0 \text{ such that} \\ 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \end{array} \right\}$$

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We can bound the function *f(n)* above by some constant multiplied by *g(n)*

For some increasing range

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$$O(n^2) = \begin{array}{lcl} f_1(x) & = & 3n^2 \\ f_2(x) & = & 1/2n^2 + 100 \\ f_3(x) & = & n^2 + 5n + 40 \\ f_4(x) & = & 6n \end{array}$$

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Generally, we're most interested in big O notation since it is an upper bound on the running time

Omega: Lower bound

- *Ω(g(n))* is the set of functions:

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Theta: Upper and lower bound

- *Θ(g(n))* is the set of functions:

$$\Theta(g(n)) = \left\{ f(n) : \begin{array}{l} \text{there exists positive constants } c_1, c_2 \text{ and } n_0 \text{ such that} \\ 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0 \end{array} \right\}$$

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We can bound the function *f(n)* above **and** below by some constant factor of *g(n)* (though different constants)

Theta: Upper and lower bound

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Note: A function is theta bounded **iff** it is big O bounded and Omega bounded

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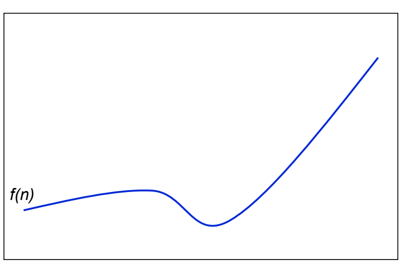
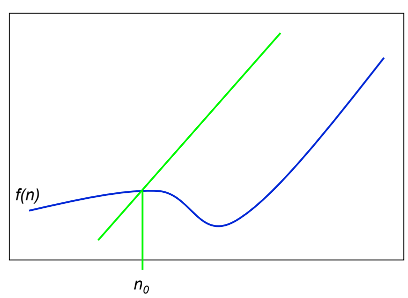
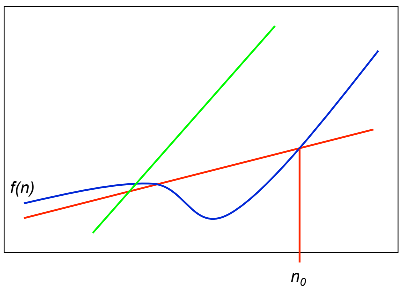
Visually



Visually: upper bound



Visually: lower bound



worst-case vs. best-case vs. average-case

- worst-case: what is the worst the running time of the algorithm can be?
- best-case: what is the best the running time of the algorithm can be?
- average-case: given random data, what is the running time of the algorithm?
- **Don't** confuse this with O, Ω and Θ. The cases above are *situations*, asymptotic notation is about bounding particular situations

Proving bounds: find constants that satisfy inequalities

- Show that $5n^2 - 15n + 100$ is $\Theta(n^2)$
- Step 1: Prove $O(n^2)$ – Find constants c and n_0 such that $5n^2 - 15n + 100 \leq cn^2$ for all $n > n_0$

$$cn^2 \geq 5n^2 - 15n + 100$$
$$c \geq 5 - 15/n + 100/n^2$$

Let $n_0 = 1$ and $c = 5 + 100 = 105$.
$100/n^2$ only get smaller as n increases and we ignore $-15/n$ since it only varies between -15 and 0

Proving bounds

- Step 2: Prove $\Omega(n^2)$ – Find constants c and n_0 such that $5n^2 - 15n + 100 \geq cn^2$ for all $n > n_0$

$$cn^2 \leq 5n^2 - 15n + 100$$
$$c \leq 5 - 15/n + 100/n^2$$

Let $n_0 = 4$ and $c = 5 - 15/4 = 1.25$ (or anything less than 1.25). We can ignore $100/n^2$ since it is always positive and $15/n$ is always decreasing.

Bounds

Is $5n^2$ $O(n)$? **No**

How would we prove it?

$$O(g(n)) = \left\{ f(n): \begin{array}{l} \text{there exists positive constants } c \text{ and } n_0 \text{ such that} \\ 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \end{array} \right\}$$

Disproving bounds

Is $5n^2$ $O(n)$?

$$O(g(n)) = \left\{ f(n): \begin{array}{l} \text{there exists positive constants } c \text{ and } n_0 \text{ such that} \\ 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \end{array} \right\}$$

Assume it's true. That means there exists some *c* and n_0 such that

$$5n^2 \le cn \text{ for } n > n_0$$

$$5n \le c$$ contradiction!

Some rules of thumb

- Multiplicative constants can be omitted
  - $14n^2$ becomes n^2
  - 7 log *n* become log *n*
- Lower order functions can be omitted
  - $n + 5$ becomes *n*
  - $n^2 + n$ becomes n^2
- n^a dominates n^b if $a > b$
  - n^2 dominates *n*, so n^2+n becomes n^2
  - $n^{1.5}$ dominates $n^{1.4}$

Some rules of thumb

- a^n dominates b^n if $a > b$
  - 3^n dominates 2^n
- **Any** exponential dominates any polynomial
  - 3^n dominates n^5
  - 2^n dominates n^c
- **Any** polynomial dominates any logorithm
  - *n* dominates log *n* or log log *n*
  - n^2 dominates *n* log *n*
  - $n^{1/2}$ dominates *log n*
- Do **not** omit lower order terms of different variables ($n^2 + m$) does not become n^2

Big O

- $n^2 + n \log n + 50$
- $2^n - 15n^2 + n^3 \log n$
- $n^{\log n} + n^2 + 15n^3$
- $n^5 - n! + n^n$

Some examples

- O(1) – constant. Fixed amount of work, regardless of the input size
  - add two 32 bit numbers
  - determine if a number is even or odd
  - sum the first 20 elements of an array
  - delete an element from a doubly linked list
- O(log *n*) – logarithmic. At each iteration, discards some portion of the input (i.e. half)
  - binary search

Some examples

- O(*n*) – linear. Do a constant amount of work on each element of the input
  - find an item in a linked list
  - determine the largest element in an array
- O(*n* log *n*) log-linear. Divide and conquer algorithms with a linear amount of work to recombine
  - Sort a list of number with MergeSort
  - FFT

Some examples

- $O(n^2)$ – quadratic. Double nested loops that iterate over the data
  - Insertion sort
- $O(2^n)$ – exponential
  - Enumerate all possible subsets
  - Traveling salesman using dynamic programming
- O(n!)
  - Enumerate all permutations
  - determinant of a matrix with expansion by minors

Divide and Conquer

- **Divide**: Break the problem into smaller subproblems
- **Conquer**: Solve the subproblems. Generally, this involves waiting for the problem to be small enough that it is trivial to solve (i.e. 1 or 2 items)
- **Combine**: Given the results of the solved subproblems, combine them to generate a solution for the complete problem

Divide and Conquer: some thoughts

- Often, the sub-problem is the same as the original problem
- Dividing the problem in half frequently does the job
- May have to get creative about how the data is split
- Splitting tends to generate run times with log *n* in them

Divide and Conquer: Sorting

- How should we split the data?
- What are the subproblems we need to solve?
- How do we combine the results from these subproblems?

MergeSort

```
Merge-Sort(A)
  if length[A] == 1
        return A
  else
        q ← ⌊length[A] /2⌋
        create arrays L[1..q] and R[q + 1.. length[A]]
        copy A[1..q] to L
        copy A[q + 1.. length[A]] to R
        LS ← Merge-Sort(L)
        RS ← Merge-Sort(R)
        return Merge(LS, RS)
```

MergeSort: Merge

- Assuming L and R are sorted already, merge the two to create a single sorted array

MERGE(L, R)

create array B of length $length[L] + length[R]$
$i \leftarrow 1$
$j \leftarrow 1$
for $k \leftarrow 1$ **to** $length[B]$
  if $j > length[R]$ **or** $(i \leq length[L]$ **and** $L[i] \leq R[j])$
    $B[k] \leftarrow L[i]$
    $i \leftarrow i + 1$
  else
    $B[k] \leftarrow R[j]$
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return B

Merge

L: 1 3 5 8   R: 2 4 6 7

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Merge

L: 1 3 5 8 R: 2 4 6 7

B:

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MERGE(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
        if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
                B[k] ← L[i]
                i ← i + 1
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Merge

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B: 1

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L: 1 3 5 8 R: 2 4 6 7

B: 1 2 3

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Merge

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B: 1 2 3 4 5

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L: 1 3 5 8 R: 2 4 6 7

B: 1 2 3 4 5 6 7

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MERGE(L, R)
  create array B of length length[L] + length[R]
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          B[k] ← L[i]
          i ← i + 1
      else
          B[k] ← R[j]
          j ← j + 1
  return B
```

Merge

L: 1 3 5 8 R: 2 4 6 7

B: 1 2 3 4 5 6 7 8

```
MERGE(L, R)
  create array B of length length[L] + length[R]
  i ← 1
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          B[k] ← L[i]
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          B[k] ← R[j]
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  return B
```

Merge

- Does the algorithm terminate?

```
MERGE(L, R)
1  create array B of length length[L] + length[R]
2  i ← 1
3  j ← 1
4  for k ← 1 to length[B]
5          if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
6                  B[k] ← L[i]
7                  i ← i + 1
8          else
9                  B[k] ← R[j]
10                 j ← j + 1
11 return B
```

Merge

- Is it correct?
  - Loop invariant:

```
MERGE(L, R)
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2  i ← 1
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Merge

- Is it correct?
  - Loop invariant: At the beginning of the **for** loop of lines 4-10 the first k-1 elements of B are the smallest *k-1* elements from L and R in sorted order.

```
MERGE(L, R)
1  create array B of length length[L] + length[R]
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Merge

- Running time?

```
MERGE(L, R)
1  create array B of length length[L] + length[R]
2  i ← 1
3  j ← 1
4  for k ← 1 to length[B]
5          if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
6                  B[k] ← L[i]
7                  i ← i + 1
8          else
9                  B[k] ← R[j]
10                 j ← j + 1
11 return B
```

Merge

- Running time? Θ(n) - linear

```
MERGE(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
        if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
                B[k] ← L[i]
                i ← i + 1
        else
                B[k] ← R[j]
                j ← j + 1
return B
```