Shortest Paths and Minimum Spanning Trees

David Kauchak
cs302
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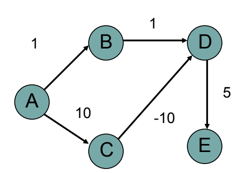
Admin

Dijkstra's algorithm

DIJKSTRA(G, s)

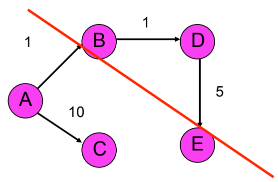
for all $v \in V$
    $dist[v] \leftarrow \infty$
    $prev[v] \leftarrow null$
$dist[s] \leftarrow 0$
$Q \leftarrow$ MAKEHEAP(V)
while !EMPTY(Q)
    $u \leftarrow$ EXTRACTMIN(Q)
    for all edges $(u, v) \in E$
        if $dist[v] > dist[u] + w(u, v)$
            $dist[v] \leftarrow dist[u] + w(u, v)$
            DECREASEKEY$(Q, v, dist[v])$
            $prev[v] \leftarrow u$

What about Dijkstra's on…?



What about Dijkstra's on…?

Dijkstra's algorithm only works for positive edge weights



Bounding the distance

- Another invariant: For each vertex v, dist[v] is an upper bound on the actual shortest distance
  - start off at ∞
  - only update the value if we find a shorter distance
- An update procedure

$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

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Can we ever go wrong applying this update rule?

- We can apply this rule as many times as we want and will never underestimate dist[v]

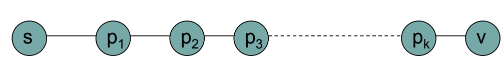
When will dist[v] be right?

- If u is along the shortest path to v and dist[u] is correct

---

$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

- dist[v] will be right if u is along the shortest path to v and dist[u] is correct
- Consider the shortest path from s to v



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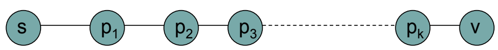
- What happens if we update all of the vertices with the above update?



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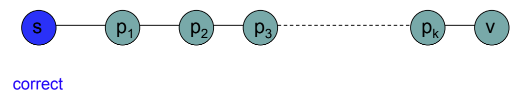

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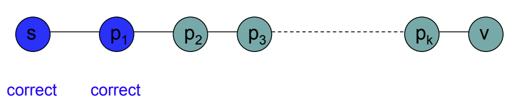

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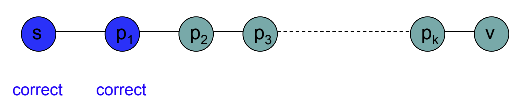
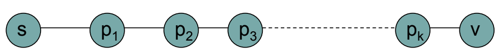
- dist[v] will be right if u is along the shortest path to v and dist[u] is correct
- Does the order that we update the vertices matter?



$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

- dist[v] will be right if u is along the shortest path to v and dist[u] is correct
- How many times do we have to do this for vertex p_i to have the correct shortest path from s?
  - i times



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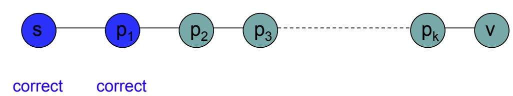

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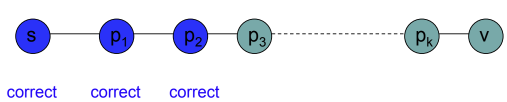

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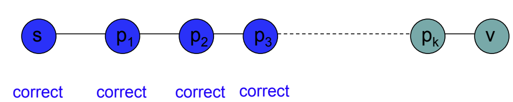

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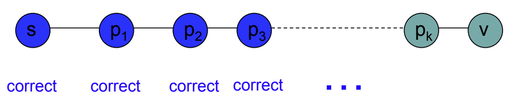
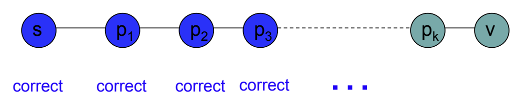


$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

- dist[v] will be right if u is along the shortest path to v and dist[u] is correct
- What is the longest (vetex-wise) the path from s to any node v can be?
  - |V| - 1 edges/vertices



Bellman-Ford algorithm

```
BELLMAN-FORD(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
for i ← 1 to |V| − 1
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
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for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
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Initialize all the distances

do it |V| -1 times

iterate over all edges/vertices and apply update rule

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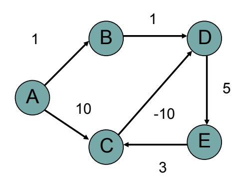
check for negative cycles

Negative cycles

What is the shortest path from a to e?



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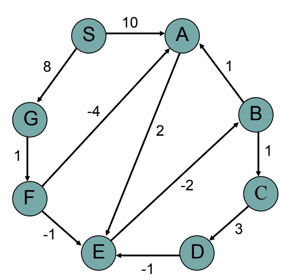
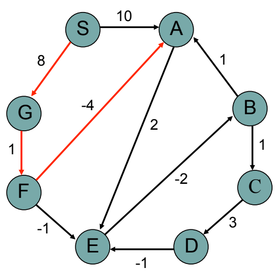
Bellman-Ford algorithm



How many edges is the shortest path from s to:

A:

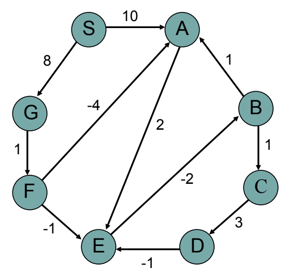
Bellman-Ford algorithm



How many edges is the shortest path from s to:

A: 3

Bellman-Ford algorithm

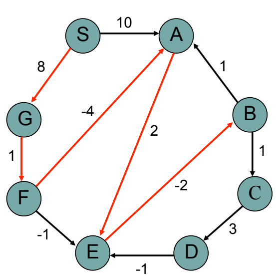


How many edges is the shortest path from s to:

A: 3

B:

Bellman-Ford algorithm

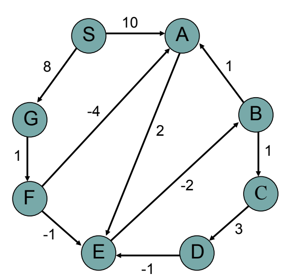


How many edges is the shortest path from s to:

A: 3

B: 5

Bellman-Ford algorithm



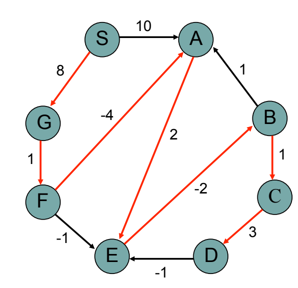
How many edges is the shortest path from s to:

A: 3

B: 5

D:

Bellman-Ford algorithm



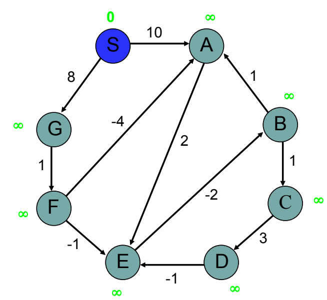
How many edges is the shortest path from s to:

A: 3

B: 5

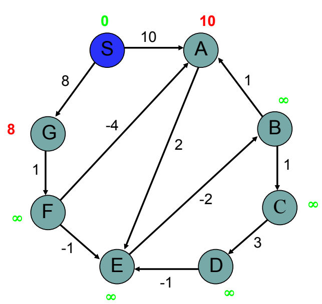
D: 7

Bellman-Ford algorithm



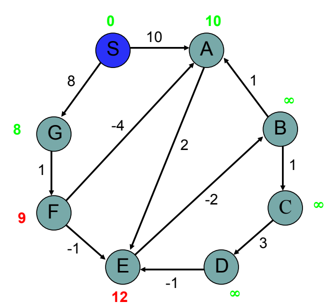
Iteration: 0

Bellman-Ford algorithm



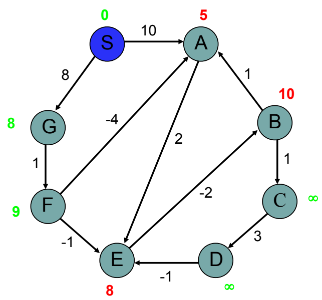
Iteration: 1

Bellman-Ford algorithm



Iteration: 2

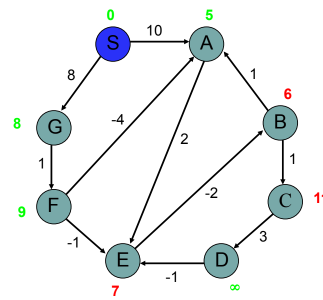
Bellman-Ford algorithm



Iteration: 3

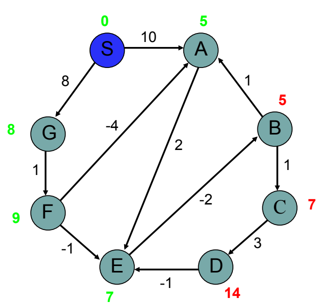
A has the correct distance and path

Bellman-Ford algorithm



Iteration: 4

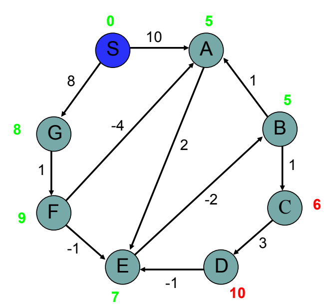
Bellman-Ford algorithm



Iteration: 5

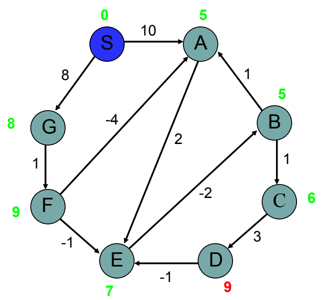
B has the correct distance and path

Bellman-Ford algorithm



Iteration: 6

Bellman-Ford algorithm



Iteration: 7

D (and all other nodes) have the correct distance and path

Correctness of Bellman-Ford

Loop invariant:

BELLMAN-FORD(G, s)
for all $v \in V$
  $dist[v] \leftarrow \infty$
  $prev[v] \leftarrow null$
$dist[s] \leftarrow 0$
for $i \leftarrow 1$ **to** $|V| - 1$
  for all edges $(u, v) \in E$
    if $dist[v] > dist[u] + w(u, v)$
      $dist[v] \leftarrow dist[u] + w(u, v)$
      $prev[v] \leftarrow u$
for all edges $(u, v) \in E$
  if $dist[v] > dist[u] + w(u, v)$
    return $false$

Correctness of Bellman-Ford

Loop invariant: After iteration i, all vertices with shortest paths from s of length i edges or less have correct distances

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Runtime of Bellman-Ford

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O(|V| |E|)

Runtime of Bellman-Ford

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for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
                return false
```

Can you modify the algorithm to run faster (in some circumstances)?

All pairs shortest paths

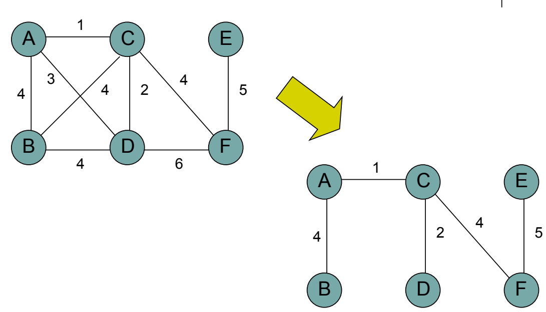
- Simple approach
  - Call Bellman-Ford |V| times
  - O(|V|^2 |E|)
- Floyd-Warshall – Θ(|V|^3)
- Johnson’s algorithm – O(|V|^2 log |V| + |V| |E|)

Minimum spanning trees

- What is the lowest weight set of edges that connects all vertices of an undirected graph with positive weights
- Input: An undirected, positive weight graph, G=(V,E)
- Output: A tree T=(V,E’) where E’ ⊆ E that minimizes

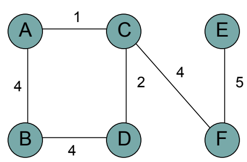
$$weight(T) = \sum_{e \in E'} w_e$$

MST example



MSTs

Can an MST have a cycle?



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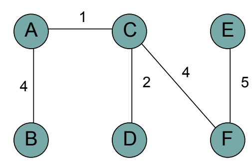
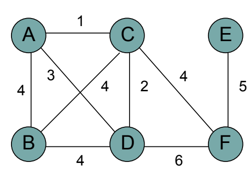
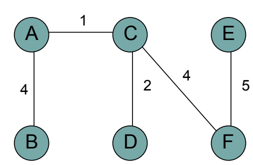


Applications?

- Connectivity
  - Networks (e.g. communications)
  - Circuit design/wiring
- hub/spoke models (e.g. flights, transportation)
- Traveling salesman problem?

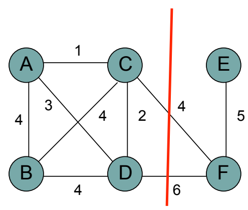
Algorithm ideas?





Cuts

- A cut is a partitioning of the vertices into two sets S and V-S
- An edge “crosses” the cut if it connects a vertex u∈V and v∈V-S



Minimum cut property

Given a partion S, let edge *e* be the minimum cost edge that **crosses** the partition. *Every* minimum spanning tree contains edge *e*.

Prove this!

Minimum cut property

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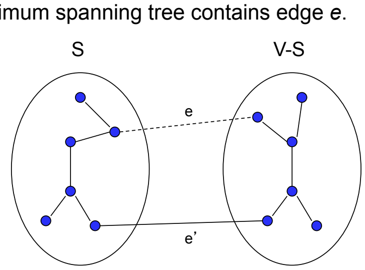

Consider an MST with edge e’ that is not the minimum edge

Minimum cut property

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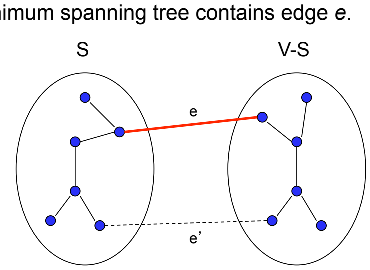


Using e instead of e’, still connects the graph, but produces a tree with smaller weights

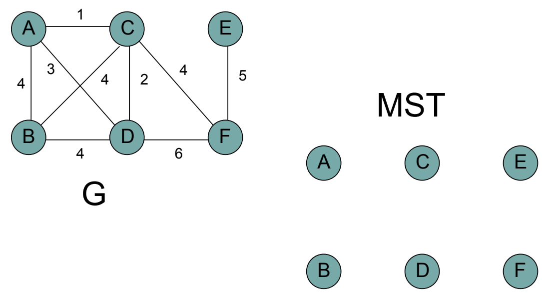
Kruskal's algorithm

Given a partition S, let edge *e* be the minimum cost edge that **crosses** the partition. *Every* minimum spanning tree contains edge *e*.

```
KRUSKAL(G)
for all v ∈ V
        MAKESET(v)
T ← {}
sort the edges of E by weight
for all edges (u, v) ∈ E in increasing order of weight
        if FIND-SET(u) ≠ FIND-SET(v)
                add edge to T
                UNION(FIND-SET(u),FIND-SET(v))
```

Kruskal's algorithm

Add smallest edge that connects two sets not already connected



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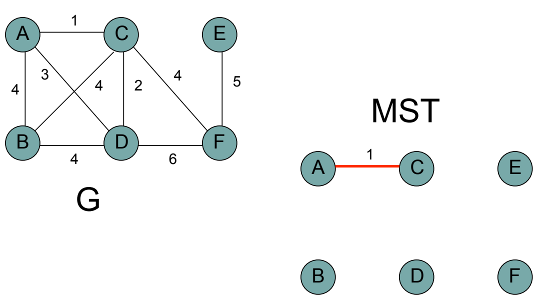

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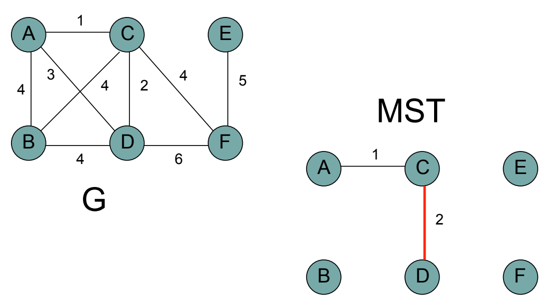

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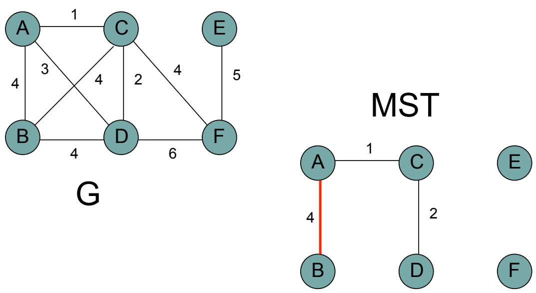

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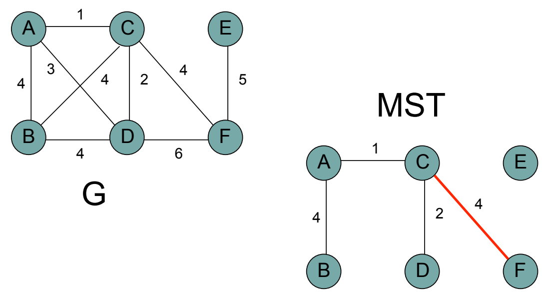
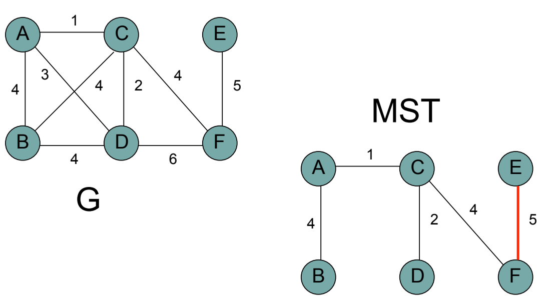
Add smallest edge that connects
two sets not already connected



Kruskal's algorithm

Add smallest edge that connects
two sets not already connected



Correctness of Kruskal's

- Never adds an edge that connects already connected vertices
- Always adds lowest cost edge to connect two sets. By min cut property, that edge must be part of the MST

```
KRUSKAL(G)
for all v ∈ V
        MAKESET(v)
T ← {}
sort the edges of E by weight
for all edges (u, v) ∈ E in increasing order of weight
        if FIND-SET(u) ≠ FIND-SET(v)
                add edge to T
                UNION(FIND-SET(u),FIND-SET(v))
```

Running time of Kruskal's

```
KRUSKAL(G)
for all v ∈ V
        MAKESET(v)
T ← {}
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        if FIND-SET(u) ≠ FIND-SET(v)
                add edge to T
                UNION(FIND-SET(u),FIND-SET(v))
```

Running time of Kruskal's

```
KRUSKAL(G)
for all v ∈ V                                            |V| calls to MakeSet
        MAKESET(v)
T ← {}
sort the edges of E by weight                            O(|E| log |E|)
for all edges (u, v) ∈ E in increasing order of weight
        if FIND-SET(u) ≠ FIND-SET(v)                     2 |E| calls to FindSet
                add edge to T
                UNION(FIND-SET(u),FIND-SET(v))           |V| calls to Union
```

Running time of Kruskal's

Disjoint set data structure

O(|E| log |E|) +

	MakeSet	FindSet |E| calls	Union |V| calls	Total
Linked lists	|V|	O(|V| |E|)	|V|	O(|V||E| + |E| log |E|) O(|V| |E|)
Linked lists + heuristics	|V|	O(|E| log |V|)	|V|	O(|E| log |V|+ |E| log |E|) O(|E| log |E|)

Prim's algorithm

```
PRIM(G, r)
for all v ∈ V
        key[v] ← ∞
        prev[v] ← null
key[r] ← 0
H ← MAKEHEAP(key)
while !Empty(H)
        u ← EXTRACT-MIN(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        DECREASE-KEY(v, w(u, v))
                        prev[v] ← u
```

Prim's algorithm

```
Prim(G, r)
for all v ∈ V
        key[v] ← ∞
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        u ← Extract-Min(H)
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        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        Decrease-Key(v, w(u, v))
                        prev[v] ← u
```

```
Dijkstra(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MakeHeap(V)
while !Empty(Q)
        u ← ExtractMin(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DecreaseKey(Q, v, dist[v])
                        prev[v] ← u
```

Prim's algorithm

```
Prim(G, r)
for all v ∈ V
        key[v] ← ∞
        prev[v] ← null
key[r] ← 0
H ← MakeHeap(key)
while !Empty(H)
        u ← Extract-Min(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        Decrease-Key(v, w(u, v))
                        prev[v] ← u
```

Prim's algorithm

Start at some root node and build out the MST by adding the lowest weighted edge at the frontier

```
Prim(G, r)
for all v ∈ V
        key[v] ← ∞
        prev[v] ← null
key[r] ← 0
H ← MakeHeap(key)
while !Empty(H)
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```

Prim's

```
while !Empty(H)
        u ← Extract-Min(H)
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```
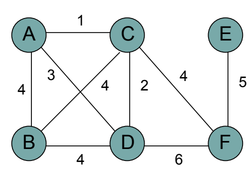
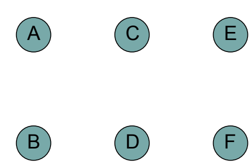
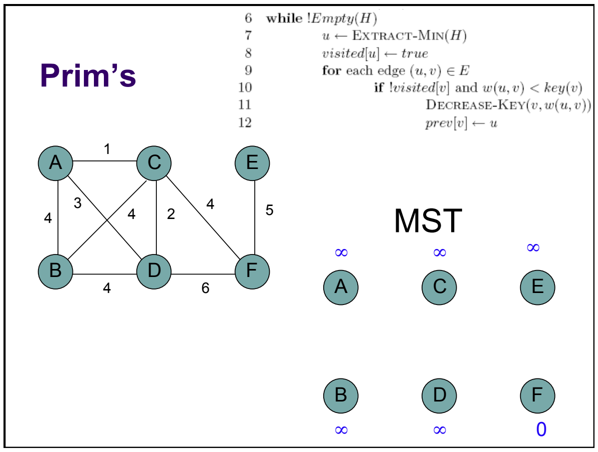



MST

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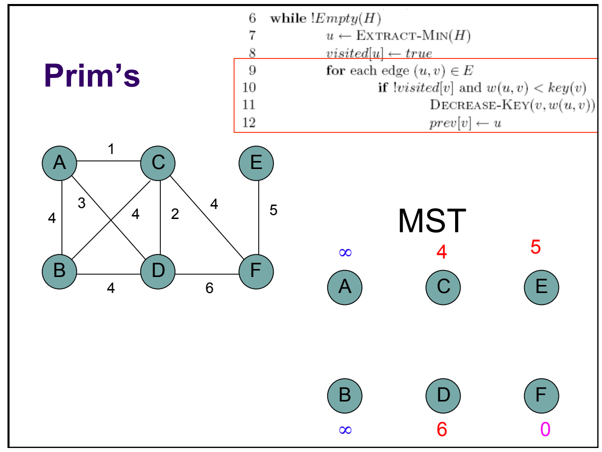
Decrease-Key(v, w(u, v))
prev[v] ← u
A C E B D F
1 3 4 4 2 4 5 4 6
MST
∞ ∞ ∞
A C E
B D F
∞ ∞ 0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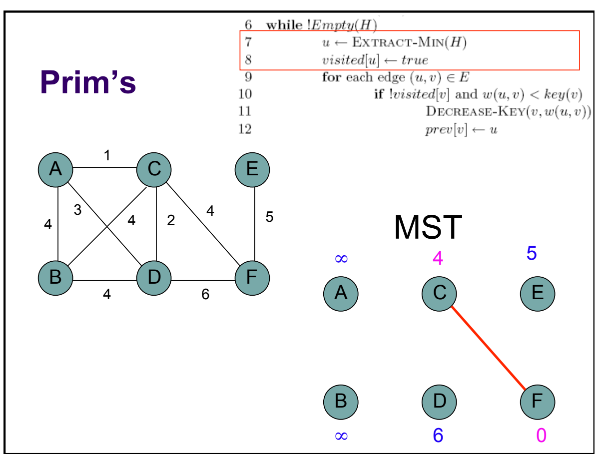
prev[v] ← u
A C E B D F
1 3 4 4 2 4 5 4 6
MST
∞ 4 5
A C E
B D F
∞ 6 0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A C E B D F
1 3 4 4 2 4 5 4 6
MST
∞ 4 5
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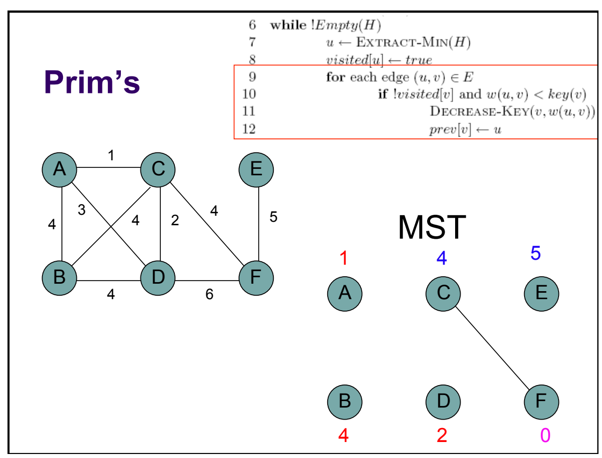
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1 3 4 4 2 4 5 4 6
MST
1 4 5
A C E
B D F
4 2 0

Prim's

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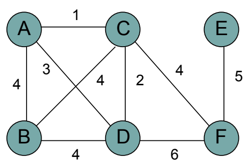

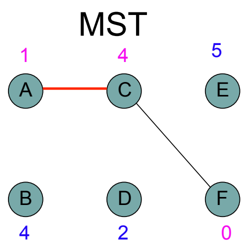

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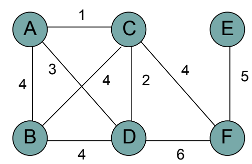

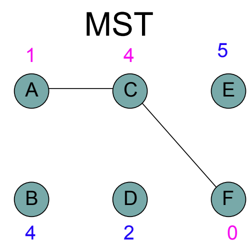

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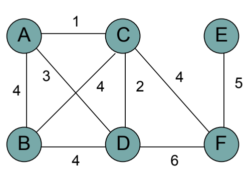

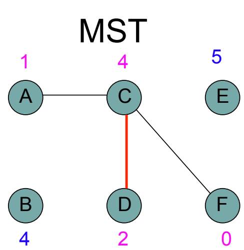

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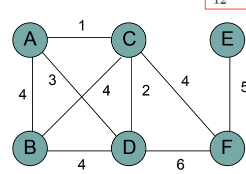

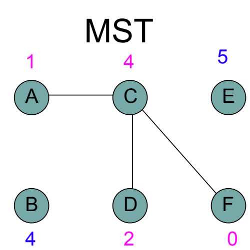

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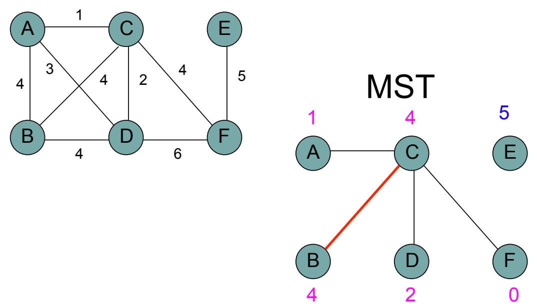

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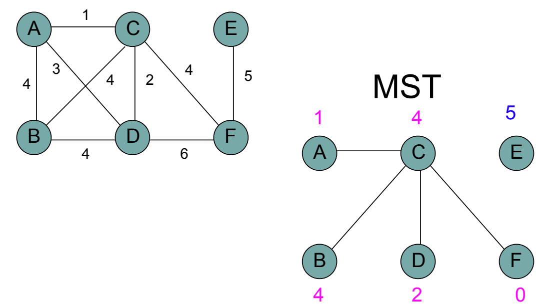

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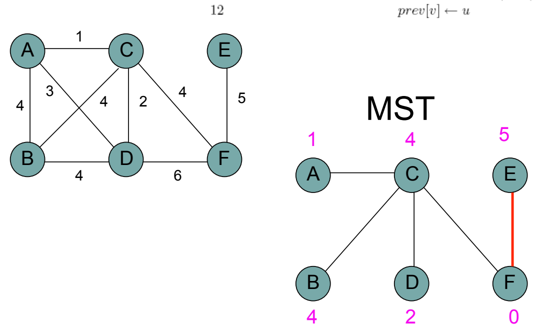

Correctness of Prim's?

- Can we use the min-cut property?
  - Given a partion S, let edge *e* be the minimum cost edge that **crosses** the partition. *Every* minimum spanning tree contains edge *e*.
- Let S be the set of vertices visited so far
- The only time we add a new edge is if it's the lowest weight edge from S to V-S

Running time of Prim's

```
PRIM(G, r)
for all v ∈ V
        key[v] ← ∞
        prev[v] ← null
key[r] ← 0
H ← MAKEHEAP(key)
while !Empty(H)
        u ← EXTRACT-MIN(H)
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```

Θ(|V|)

Θ(|V|)

|V| calls to Extract-Min

|E| calls to Decrease-Key

Running time of Prim's

Same as Dijksta's algorithm

	1 MakeHeap	|V| ExtractMin	|E| DecreaseKey	Total
Array	O(|V|)	O(|V|^2)	O(|E|)	O(|V|^2)
Bin heap	O(|V|)	O(|V| log |V|)	O(|E| log |V|)	O((|V|+|E|) log |V|) O(|E| log |V|)
Fib heap	O(|V|)	O(|V| log |V|)	O(|E|)	**O(|V| log |V| + |E|)**

Kruskal's: O(|E| log |E|)