Hashtables

David Kauchak
cs302
Spring 2012

Administrative

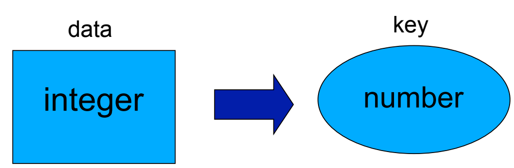
- Midterm
  - must take it by Friday at 6pm
- No assignment over the break

Hashtables

- Constant time insertion and search (and deletion in some cases) for a large space of keys
- Applications
  - Does *x* belong to *S?*
  - I've found them very useful
  - compilers
  - databases
  - search engines
  - storing and retrieving non-sequential data
  - save memory over an array

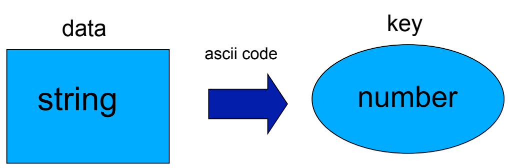
Key/data pair

- The key is a numeric representation of a *relevant portion* of the data
- For example:



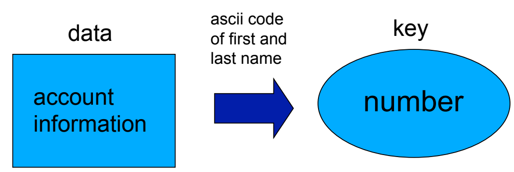
Key/data pair

- The key is a numeric representation of a *relevant portion* of the data
- For example:

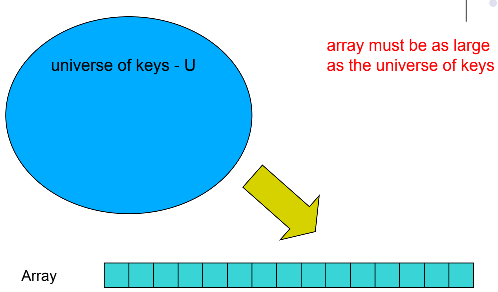


Key/data pair

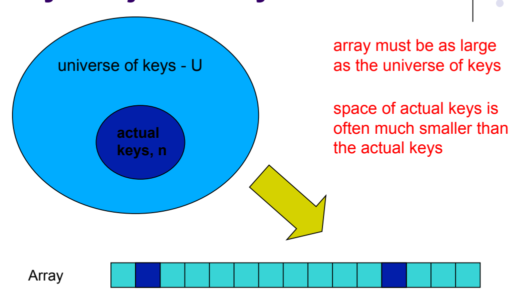
- The key is a numeric representation of a *relevant portion* of the data
- For example:



Why not just arrays aka direct-address tables?



Why not just arrays?



Why not arrays?

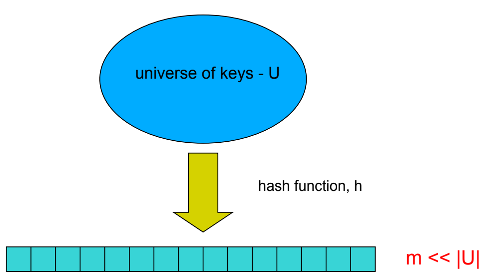
- Think of indexing all last names < 10 characters
  - Census listing of all last names
    http://www.census.gov/genealogy/names/dist.all.last
    - 88,799 last names
  - What is the size of our space of keys?
    - 26^{10} = a big number
  - Not feasible!
  - Even if it were, not space efficient

The load of a table/hashtable

- m = number of possible entries in the table
- n = number of keys stored in the table
- α = n/m is the **load factor** of the hashtable
- What is the load factor of the last example?
  - α = 88,799 / 26^{10} would be the load factor of last names using direct-addressing
- The smaller α, the more wasteful the table
- The load also helps us talk about run time

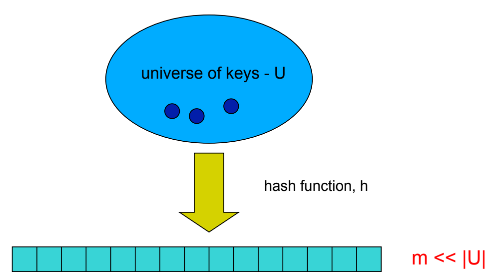
Hash function, *h*

- A hash function is a function that maps the universe of keys to the slots in the hashtable



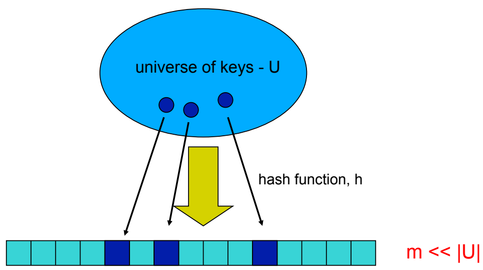
Hash function, *h*

- A hash function is a function that maps the universe of keys to the slots in the hashtable



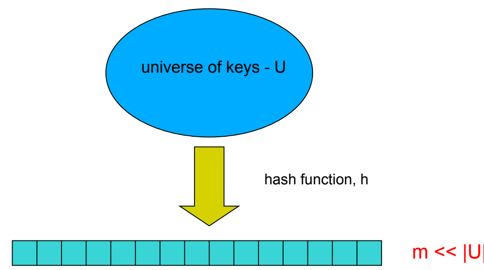
Hash function, *h*

- A hash function is a function that maps the universe of keys to the slots in the hashtable



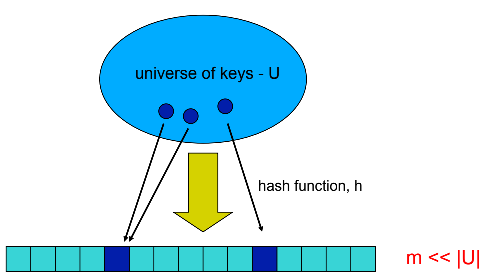
Hash function, *h*

- What can happen if m ≠ |U|?



Collisions

- If m ≠ |U|, then two keys can map to the same position in the hashtable (pidgeonhole principle)

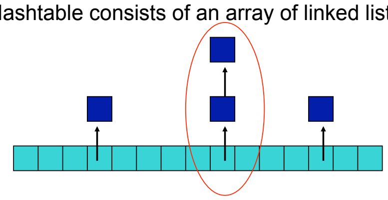


Collisions

- A collision occurs when h(x) = h(y), but x ≠ y
- A good hash function will minimize the number of collisions
- Because the number of hashtable entries is less than the possible keys (i.e. m < |U|) collisions are inevitable!
- Collision resolution techniques?

Collision resolution by chaining

- Hashtable consists of an array of linked lists



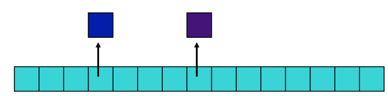
- When a collision occurs, the element is added to linked list at that location
- If two entries x ≠ y have the same hash value h(x) = h(x), then T(h(x)) will contain a linked list with both values

Insertion

$\textsc{ChainedHashInsert}(T, x)$

  insert x at the head of list $T[h(x)]$

ChainedHashInsert( )



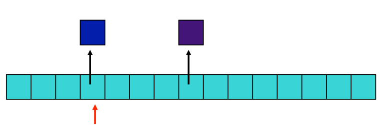
Insertion

$\textsc{ChainedHashInsert}(T, x)$

  insert x at the head of list $T[h(x)]$

h( )

hash function is a mapping from the key to some value < m

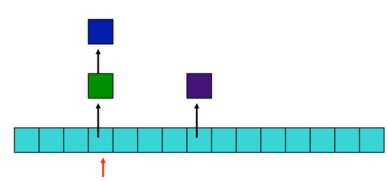


Insertion

$\textsc{ChainedHashInsert}(T, x)$

  insert x at the head of list $T[h(x)]$

h( )



Deletion

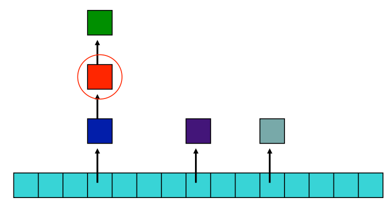
ChainedHashInsert(T, x)
  insert x at the head of list $T[h(x)]$

x is a reference not the value, why?

Remember, we're hashing based on a numeric representation of the actual underlying data

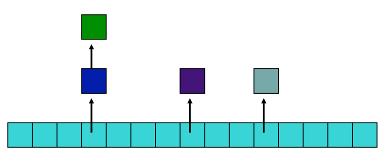
Deletion

ChainedHashDelete(T, x)
  delete x from the list $T[h(key[x])]$



Deletion

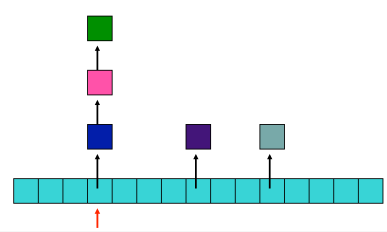
ChainedHashDelete(T, x)
  delete x from the list $T[h(key[x])]$



Search

ChainedHashSearch(T, x)
  search for x in list $T[h(x)]$

ChainedHashSearch( )

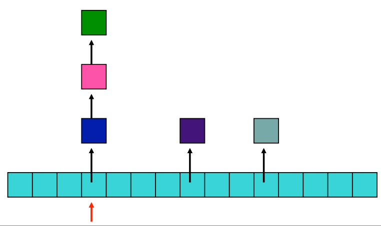


Search

$\text{ChainedHashSearch}(T, x)$

search for x in list $T[h(x)]$

h(■)

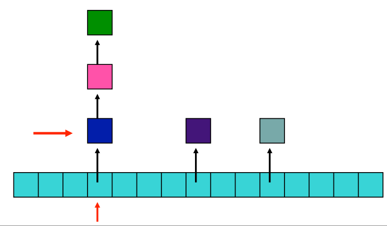


Search

$\text{ChainedHashSearch}(T, x)$

search for x in list $T[h(x)]$

ChainedHashSearch(■)

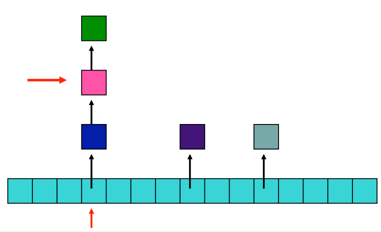


Search

$\text{ChainedHashSearch}(T, x)$

search for x in list $T[h(x)]$

ChainedHashSearch(■)

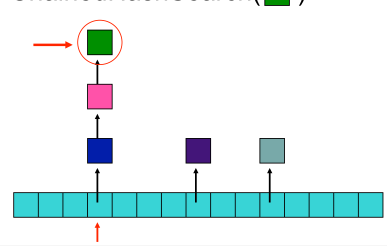


Search

$\text{ChainedHashSearch}(T, x)$

search for x in list $T[h(x)]$

ChainedHashSearch(■)



Running time

CHAINEDHASHINSERT(T, x)
insert x at the head of list $T[h(x)]$

$\Theta(1)$

CHAINEDHASHDELETE(T, x)
delete x from the list $T[h(key[x])]$

O(length of the chain)

CHAINEDHASHSEARCH(T, x)
search for x in list $T[h(x)]$

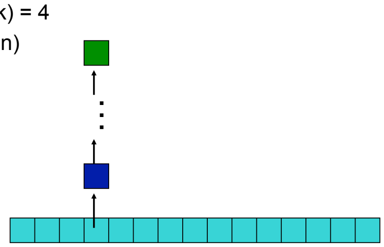
O(length of the chain)

Length of the chain

- Worst case?

Length of the chain

- Worst case?
  - All elements hash to the same location
  - h(k) = 4
  - O(n)



Length of the chain

- Average case
  - Depends on how well the hash function distributes the keys
  - What is the best we could hope for a hash function?
    - simple uniform hashing: an element is equally likely to end up in any of the m slots
  - Under simple uniform hashing what is the average length of a chain in the table?
    - n keys over m slots = $n / m = \alpha$

Average chain length

- If you roll a fair *m* sided die *n* times, how many times are we likely to see a given value?
- For example, 10 sided die:
  - 1 time
    - 1/10
  - 100 times
    - 100/10 = 10

Search average running time

- Two cases:
  - Key is **not** in the table
    - must search all entries
    - $\Theta(1 + \alpha)$
  - Key **is** in the table
    - on average search half of the entries
    - $O(1 + \alpha)$

Hash functions

- What makes a good hash function?
  - Approximates the assumption of simple uniform hashing
  - Deterministic – h(x) should always return the same value
  - Low cost – if it is expensive to calculate the hash value (e.g. log n) then we don't gain anything by using a table
- Challenge: we don't generally know the distribution of the keys
  - Frequently data tend to be clustered (e.g. similar strings, run-times, SSNs). A good hash function should spread these out across the table

Hash functions

What are some hash functions you've heard of before?

Division method

- h(k) = k mod m

m	k	h(k)
11	25	3
11	1	1
11	17	6
13	133	3
13	7	7
13	25	12

Division method

- **Don't** use a power of two. Why?

m	k	bin(k)	h(k)
8	25	11001	1
8	1	00001	1
8	17	10001	1

- if h(k) = k mod 2^p, the hash function is just the lower *p* bits of the value

Division method

- Good rule of thumb for *m* is a prime number not to close to a power of 2
- Pros:
  - quick to calculate
  - easy to understand
- Cons:
  - keys close to each other will end up close in the hashtable

Multiplication method

- Multiply the key by a constant 0 < A < 1 and extract the fractional part of *kA*, then scale by *m* to get the index

$$h(k) = \lfloor m(kA - \lfloor kA \rfloor) \rfloor$$

extracts the fractional portion of *kA*

Multiplication method

$$h(k) = \lfloor m(kA - \lfloor kA \rfloor) \rfloor$$

- Common choice is for *m* as a power of 2 and

$$A = (\sqrt{5} - 1)/2 = 0.6180339887$$

- Why a power of 2?
- Book has other heuristics

Multiplication method

m	k	A	kA	h(k)
8	15	0.618	9.27	floor(0.27*8) = 2
8	23	0.618	14.214	floor(0.214*8) = 1
8	100	0.618	61.8	floor(0.8*8) = 6

$$h(k) = \lfloor m(kA - \lfloor kA \rfloor) \rfloor$$

Other hash functions

- http://en.wikipedia.org/wiki/List_of_hash_functions
- cyclic redundancy checks (i.e. disks, cds, dvds)
- Checksums (i.e. networking, file transfers)
- Cryptographic (i.e. MD5, SHA)

Open addressing

- Keeping around an array of linked lists can be inefficient and a hassle
- Like to keep the hashtable as just an array of elements (no pointers)
- How do we deal with collisions?
  - compute another slot in the hashtable to examine

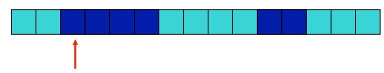
Hash functions with open addressing

- Hash function must define a **probe sequence** which is the list of slots to examine when searching or inserting
- Hash function takes an additional parameter *i* which is the number of collisions that have already occurred
- The probe sequence **must** be a permutation of every hashtable entry. Why?

{ h(k,0), h(k,1), h(k,2), …, h(k, m-1) } is a permutation of { 0, 1, 2, 3, …, m-1 }

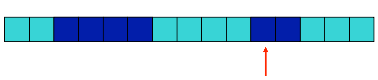
Probe sequence

h(k, 0)



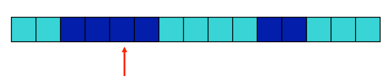
Probe sequence

h(k, 1)



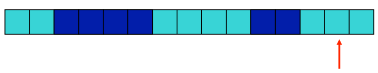
Probe sequence

h(k, 2)



Probe sequence

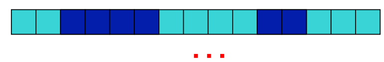
h(k, 3)



Probe sequence

h(k, …)

must visit all locations



Open addressing: Insert

```
Hash-Insert(T, k)
i ← 0
j ← h(k, i)
while i < m − 1 and T[j] ≠ null
        i ← i + 1
        j ← h(k, i)
if T[j] = null
        return j
else
        error "hash is full"
```

Open addressing: Insert

```
Hash-Insert(T, k)
i ← 0
j ← h(k, i)
while i < m − 1 and T[j] ≠ null
        i ← i + 1
        j ← h(k, i)
if T[j] = null
        return j
else
        error "hash is full"
```

get the first hashtable entry to look in

Open addressing: Insert

HASH-INSERT(T, k)

$i \leftarrow 0$
$j \leftarrow h(k, i)$
while $i < m - 1$ and $T[j] \neq \mathit{null}$
    $i \leftarrow i + 1$
    $j \leftarrow h(k, i)$
if $T[j] = \mathit{null}$
    return j
else
    error "hash is full"

follow the probe sequence until we find an open entry

Open addressing: Insert

HASH-INSERT(T, k)

$i \leftarrow 0$
$j \leftarrow h(k, i)$
while $i < m - 1$ and $T[j] \neq \mathit{null}$
    $i \leftarrow i + 1$
    $j \leftarrow h(k, i)$
if $T[j] = \mathit{null}$
    return j
else
    error "hash is full"

return the open entry

Open addressing: Insert

HASH-INSERT(T, k)

$i \leftarrow 0$
$j \leftarrow h(k, i)$
while $i < m - 1$ and $T[j] \neq \mathit{null}$
    $i \leftarrow i + 1$
    $j \leftarrow h(k, i)$
if $T[j] = \mathit{null}$
    return j
else
    error "hash is full"

hashtable can fill up

Open addressing: search

HASH-SEARCH(T, k)

$i \leftarrow 0$
$j \leftarrow h(k, i)$
while $i < m - 1$ and $T[j] \neq \mathit{null}$ and $T[j] \neq k$
    $i \leftarrow i + 1$
    $j \leftarrow h(k, i)$
if $T[j] = k$
    return j
else
    return null

Open addressing: search

Hash-Search(T, k)
$i \leftarrow 0$
$j \leftarrow h(k, i)$
while $i < m - 1$ and $T[j] \neq null$ and $T[j] \neq k$
  $i \leftarrow i + 1$
  $j \leftarrow h(k, i)$
if $T[j] = k$
  return j
else
  return *null*

Hash-Insert(T, k)
$i \leftarrow 0$
$j \leftarrow h(k, i)$
while $i < m - 1$ and $T[j] \neq null$
  $i \leftarrow i + 1$
  $j \leftarrow h(k, i)$
if $T[j] = null$
  return j
else
  error "hash is full"

Open addressing: search

Hash-Search(T, k)
$i \leftarrow 0$
$j \leftarrow h(k, i)$
while $i < m - 1$ and $T[j] \neq null$ and $T[j] \neq k$
  $i \leftarrow i + 1$
  $j \leftarrow h(k, i)$
if $T[j] = k$
  return j
else
  return *null*

Hash-Insert(T, k)
$i \leftarrow 0$
$j \leftarrow h(k, i)$
while $i < m - 1$ and $T[j] \neq null$
  $i \leftarrow i + 1$
  $j \leftarrow h(k, i)$
if $T[j] = null$
  return j
else
  error "hash is full"

"breaks" the probe sequence

Open addressing: delete

- Two options:
  - mark node as "deleted" (rather than null)
    - modify search procedure to continue looking if a "deleted" node is seen
    - modify insert procedure to fill in "deleted" entries
    - increases search times
  - if a lot of deleting will happen, use chaining

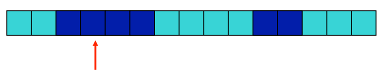
Probing schemes

- Linear probing – if a collision occurs, go to the next slot
  - h(k,i) = (h(k) + i) mod m
  - Does it meet our requirement that it visits every slot?
  - for example, m = 7 and h(k) = 4

h(k,0) = 4
h(k,1) = 5
h(k,2) = 6
h(k,3) = 0
h(k,3) = 1

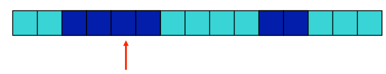
Linear probing: search

h(■, 0)



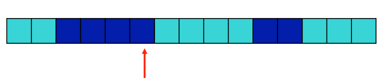
Linear probing: search

h(■, 1)



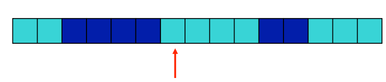
Linear probing: search

h(■, 2)



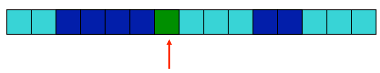
Linear probing: search

h(■, 3)



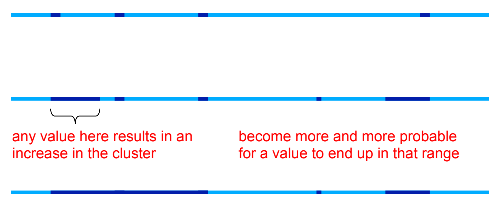
Linear probing: search

h(, 3)



Linear probing

- Problem:
  - primary clustering – long rungs of occupied slots tend to build up and these tend to grow



Quadratic probing

- $h(k,i) = (h(k) + c_1 i + c_2 i^2) \bmod m$
- Rather than a linear sequence, we probe based on a quadratic function
- Problems:
  - must pick constants and *m* so that we have a proper probe sequence
  - if h(x) = h(y), then h(x,i) = h(y,i) for all i
  - secondary clustering

Double hashing

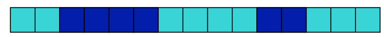
- Probe sequence is determined by a second hash function
- $h(k,i) = (h_1(k) + i(h_2(k))) \bmod m$
- Problem:
  - $h_2(k)$ must visit all possible positions in the table

Running time of insert and search for open addressing

- Depends on the hash function/probe sequence
- Worst case?
  - O(n) – probe sequence visits every full entry first before finding an empty

Running time of insert and search for open addressing

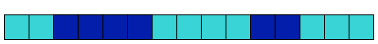
- Average case?
- We have to make at least one probe



Running time of insert and search for open addressing

- Average case?
- What is the probability that the first probe will **not** be successful (assume uniform hashing function)?

α

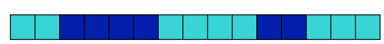


Running time of insert and search for open addressing

- Average case?
- What is the probability that the first **two** probed slots will **not** be successful?

why '~'?

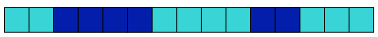
$\sim\alpha^2$



Running time of insert and search for open addressing

- Average case?
- What is the probability that the first **three** probed slots will **not** be successful?

~α^3



Running time of insert and search for open addressing

- Average case: expected number of probes
  - sum of the probability of making 1 probe, 2 probes, 3 probes, …

$$E[probes] = 1 + \alpha + \alpha^2 + \alpha^3 + \ldots$$
$$= \sum_{i=0}^{m} \alpha^i$$
$$< \sum_{i=0}^{\infty} \alpha^i$$
$$= \frac{1}{1-\alpha}$$

Average number of probes

$$E[probes] = \frac{1}{1-\alpha}$$

α	Average number of searches
0.1	$1/(1-.1) = 1.11$
0.25	$1/(1-.25) = 1.33$
0.5	$1/(1-.5) = 2$
0.75	$1/(1-.75) = 4$
0.9	$1/(1-.9) = 10$
0.95	$1/(1-.95) = 20$
0.99	$1/(1-.99) = 100$

How big should a hashtable be?

- A good rule of thumb is the hashtable should be around half full
- What happens when the hashtable gets full?
  - Copy: Create a new table and copy the values over
    - results in one expensive insert
    - simple to implement
  - Amortized copy: When a certain ratio is hit, grow the table, but copy the entries over a few at a time with every insert
    - no single insert is expensive and can guarantee per insert performance
    - more complicated to implement