csci54 – discrete math & functional programming
proofs on sets, functions

Recall: sets

- A set is an unordered collection of objects
- Given a set S and an object o, either $o \in S$ or $o \notin S$
- The cardinality of a set is written $|S|$ and is the number of elements in the set
- Special sets:
  - the empty set, which contains no elements: {}
  - the universal set, U
- Set operations: complement, union, intersection, difference, Cartesian product
- Comparing/relating sets: equality, subset, proper subset, superset, proper superset, disjoint

Set operations

- S^c: set complement
  - $S^c = \{ x \in U : x \notin S \}$
- $S \cup T$: set union
  - $S \cup T = \{ x : x \in S \text{ or } x \in T \}$
- $S \cap T$: set intersection
  - $S \cap T = \{ x : x \in S \text{ and } x \in T \}$
- S-T: set difference
  - $S - T = \{ x : x \in S \text{ and } x \notin T \}$
- SxT: Cartesian product
  - $A \times B = \{ (x,y) : x \in A \text{ and } y \in B \}$

Comparing/relating sets

- $=$: set equality
- S and T contain the same elements
- $\subseteq$: subset
- S contains T
- $\subset$: proper subset
- S contains T and S does not equal T
- $\supseteq$: superset
- T contains S
- $\supset$: proper superset
- T contains S and T does not equal S
- "T and S are disjoint"
- T and S share no elements

proofs on sets

- Element-wise:
  - Show that no matter which elements of the sets are picked, membership/non-membership is provable
- Algebraic:
  - Use properties of the operations to show relations between sets

proofs on sets

- Claim $\{x \in \mathbb{Z} : 18|x\} \subseteq \{x \in \mathbb{Z} : 6|x\}$

let y be an arbitrary element of the set $\{x \in \mathbb{Z} : 18|x\}$
then direct proof using definition of subset

- Proof:
  - Let y be an arbitrary element of the set $\{x \in \mathbb{Z} : 18|x\}$
  - This means there exists an integer k such that $y = 18k$
  - Furthermore, since $y = 18k = 6(3k)$, we know that $6|y$ and $y \in \{x \in \mathbb{Z} : 6|x\}$
  - Since this is true for any element $y \in \{x \in \mathbb{Z} : 18|x\}$, it is true for every element.
  - Therefore $\{x \in \mathbb{Z} : 18|x\} \subseteq \{x \in \mathbb{Z} : 6|x\}$

proofs on sets

- Claim $\{x \in \mathbb{Z} : 18|x\} \subseteq \{x \in \mathbb{Z} : 6|x\}$

- Is the claim still true if we replace subset with strict subset? Prove your answer correct
- Is the claim still true if we replace subset with equals? Prove your answer correct
- More generally:
  - how do you prove set equality?
    - prove subset in both directions
  - how do you prove strict subset?
    - prove subset and not equal

Digression

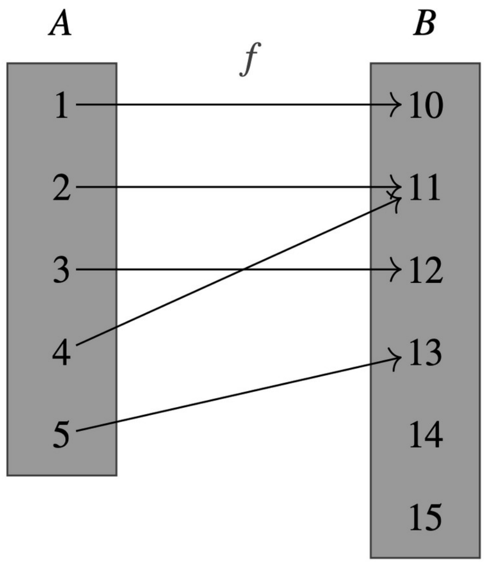
- Early next week the CS pre-pre-enrollment survey will go out!
  - To continue taking CS courses, fill this out ASAP
- Declaring the CS major:
  - Are you an on-cycle student? (Exp. grad in a Spring semester)
    - Declare halfway through CS62 next semester
  - Are you an off-cycle sophomore?
    - You need 62 + 101 next semester and to declare now
- CS minor/just taking courses
  - Fill out pre-pre-enrollment and relax

Digression

- Around pre-registration time, look out for TA interest form
  - (Ca. second week of November)
- Talk to your advisor over the next week or two to get registration clearance
- Reminder about CS sequence:
  - 51,54,62
  - 101,105,140
  - Electives

Recall: functions

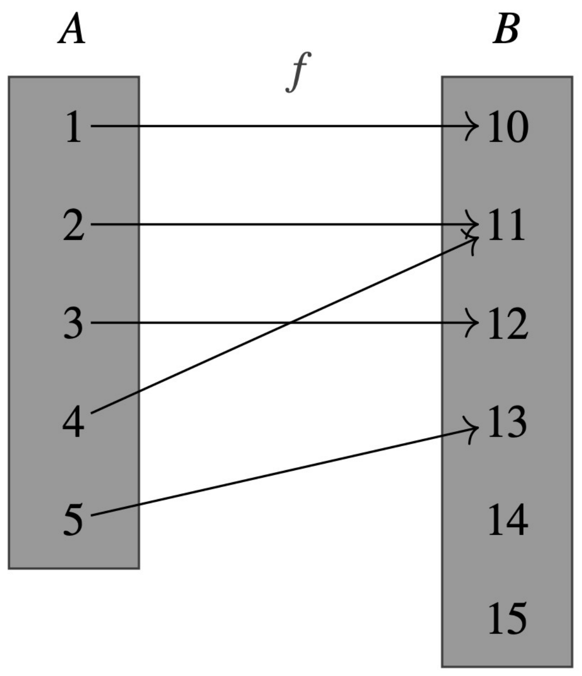
Definition 2.46: Function.
Let A and B be sets. A *function f from A to B*, written $f : A \to B$, assigns to each input value $a \in A$ a unique output value $b \in B$; the unique value b assigned to a is denoted by $f(a)$. We sometimes say that *f maps a to $f(a)$*.



- Given a function
  - the domain is the set A
  - the co-domain is the set B
  - the range (or the image) is the subset of B that are actually mapped to by an element in A.

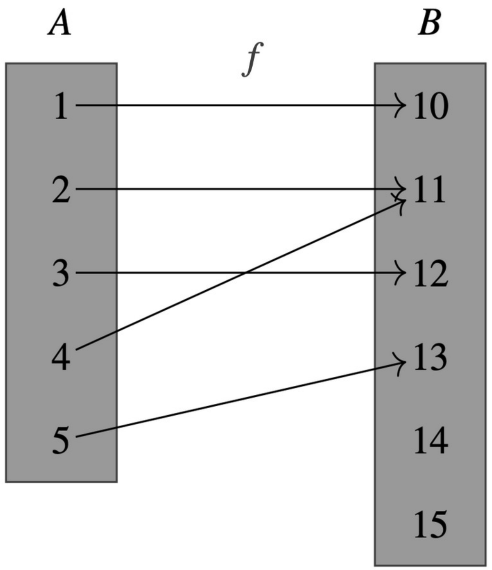
classifying functions - definitions

- <u>one-to-one</u>: a function is one-to-one if, for every element of the co-domain, at most one element of the domain maps to it.
- <u>onto</u>: a function is onto if, for every element of the co-domain, there is an element of the domain that maps to it.
  - alternatively, a function is onto if the co-domain equals the range
- <u>bijection</u>: a function is a bijection if it is both one-to-one and onto



classifying functions

- one-to-one: a function is one-to-one if, for every element of the co-domain, at most one element of the domain maps to it.
  - in other words: if f(x) = f(y), then x=y.



- onto: a function is onto if, for every element of the co-domain, there is an element of the domain that maps to it.
  - in other words, f(y) = x

example

- Claim: the function g(x) = x-1 is a bijection

a function is a bijection if it is both one-to-one and onto

- Proof:
  - g is one-to-one:

    one-to-one: if f(x) = f(y), then x=y

    - assume there are two elements x, y in Z such that g(x)=g(y) . Then x-1= y-1, so x=y
    - therefore g is one-to-one

    onto: f(y) = x

  - g is onto:
    - let x be any element of Z. Then x+1 is an element that maps to x.
    - since x is any element of Z, every element of Z has an element that maps to it,
    - therefore g is onto
  - since g is one-to-one and onto, g is a bijection

a little more on bijections

- If a function is a bijection, then it is also <u>invertible</u>. In other words, if f is a bijection, then there is a function f^{-1} such that $f(x) = y$ iff $f^{-1}(y) = x$

- The <u>identity function</u> $f(x)=x$ is a bijection
  - the identity function is the function that maps every element to itself