## csci54 - discrete math \& functional programming recurrence relations, (strong) induction

## looking ahead

- this week:
- week09-group (review)
- week09-ps
- next week:
- Tuesday: review (depending on week09-group responses)
- Thursday: checkpoint in class
- reminder to schedule proctored exam with SDRC if have accommodations:
https://www.pomona.edu/accessibility/student-accessibility/accommodation-polici es-and-procedures/test-accommodations


## proofs

- logic
- proof techniques so far
- direct proofs
- proof of the contrapositive
- proof by example / disproof by counterexample
- using cases
- induction
- today:
- more induction, including strong induction (for when regular induction is not enough)


## Proofs by induction

## Definition 5.1: Proof by mathematical induction.

Suppose that we want to prove that $P(n)$ holds for all $n \in \mathbb{Z} \geq 0$. To give a proof by mathematical induction of $\forall n \in \mathbb{Z} \geq 0: P(n)$, we prove two things:

1 the base case: prove $P(0)$.
2 the inductive case: for every $n \geq 1$, prove $P(n-1) \Rightarrow P(n)$.

- we prove the claim using a proof by induction on:
- base case:
- inductive hypothesis (IHOP):
- inductive step:
- therefore by the principle of mathematical induction:


## Recurrence relations

- Consider Sierpinski's triangle.

- Let $\mathrm{T}(\mathrm{n})$ be the number of filled triangles in a Sierpinski's triangle after $n$ iterations, where $T(0)$ is a single filled triangle.
- Observation: $T(n)=3 T(n-1)$ where $T(0)=1$
- Claim: $T(n)=3^{n}$


## Recurrence relations

- A function that is defined in terms of itself
- How would you prove that $A(n)$ is odd for all $N$ for the following recurrence rela

$$
\begin{aligned}
& A(n)=A(n-1)+A(n-2)+A(n-3) \\
& A(0)=1 \\
& A(1)=1 \\
& A(2)=3
\end{aligned}
$$

- base case: $A(0)$ is odd
- IHOP: $A(n)$ is odd
- inductive step: wts $A(n+1)$ is odd
- $A(n+1)=A(n)+A(n-1)+A(n-2)$
- $A(n)$ is odd
- now what?


## Proofs by strong induction

## Definition 5.1: Proof by mathematical induction.

Suppose that we want to prove that $P(n)$ holds for all $n \in \mathbb{Z} \geq 0$. To give a proof by mathematical induction of $\forall n \in \mathbb{Z}^{\geq 0}: P(n)$, we prove two things:

1 the base case: prove $P(0)$.
2 the inductive case: for every $n \geq 1$, prove $P(n-1) \Rightarrow P(n)$.

## Definition 5.10: Proof by strong induction.

Suppose that we want to prove that $P(n)$ holds for all $n \in \mathbb{Z} \geq 0$. To give a proof by strong induction of $\forall n \in \mathbb{Z}^{\geq 0}: P(n)$, we prove the following:

1 the base case: prove $P(0)$.
2 the inductive case: for every $n \geq 1$, prove $[P(0) \wedge P(1) \wedge \cdots \wedge P(n-1)] \Rightarrow P(n)$.

## Proofs by strong induction

## Definition 5.10: Proof by strong induction.

Suppose that we want to prove that $P(n)$ holds for all $n \in \mathbb{Z} \geq 0$. To give a proof by strong induction of $\forall n \in \mathbb{Z}^{\geq 0}: P(n)$, we prove the following:

1 the base case: prove $P(0)$.
2 the inductive case: for every $n \geq 1$, prove $[P(0) \wedge P(1) \wedge \cdots \wedge P(n-1)] \Rightarrow P(n)$.

- we prove the claim using a proof by strong induction on:
- base case(s):
may need more than one base case; need for every n where
- inductive hypothesis (IHOP):
- inductive step:

```
IHOP: assume true for all values up to n-1
```

- wts:
- therefore by the principle of mathematical inductiond: ${ }^{\text {divative step: wts true for } n}$


## Proofs by strong induction

$$
\begin{aligned}
& A(n)=A(n-1)+A(n-2)+A(n-3) \\
& A(0)=1 \\
& A(1)=1 \\
& A(2)=3
\end{aligned}
$$

## claim: $\mathrm{A}(\mathrm{n})$ is odd for all N

- we prove the claim using a proof by strong induction on:
- base case(s):
- inductive hypothesis (IHOP):
- inductive step:
- wts:
- therefore by the principle of mathematical induction:


## Proofs by strong induction

$$
\begin{aligned}
& A(n)=A(n-1)+A(n-2)+A(n-3) \\
& A(0)=1 \\
& A(1)=1 \\
& A(2)=3
\end{aligned}
$$

## Claim: $A(n)$ is odd for all N

- we prove the claim using a proof by strong induction on n
- base case(s): $A(0), A(1)$, and $A(2)$ are all odd.
- inductive hypothesis (IHOP): $A(x)$ is odd for all $x<y$
- inductive step: we want to show that $A(y)$ is odd
- by the IHOP we know that $A(y-1), A(y-2), A(y-3)$ are all odd, so there exist integers a,b,c such that ...
- this means $a+b+c$ is odd and therefore $A(y)$ is also odd.
- therefore by the principle of mathematical induction: $A(n)$ is odd


## Strong vs. regular (weak) induction

- Anything that can be proven using regular induction can also be shown using strong induction.
- However, if you can prove something using regular induction, you should.

