$$
\begin{array}{r}
\text { csci54 - discrete math \& functional programming } \\
\text { induction }
\end{array}
$$

## this week: continuing with proofs

- logic
- propositional
- predicate
- proof techniques
- direct proofs
- proof of the contrapositive
- proof by example / disproof by counterexample
- proof by contradiction
- using cases
- induction!


## some definitions (recap)

- an integer $k$ is even if and only if there exists an integer $r$ such that $k=2 r$
- an integer $k$ is odd if and only if there exists an integer $r$ such that $k=2 r+1$
- $k \mid m$ if and only if there exists an integer $r$ such that $m=k r$. This is equivalent to saying that " $m$ mod $k=0$ " or that " $k$ evenly divides $\mathrm{m} "$.
- an integer $k>1$ is prime if the only positive integers that evenly divide $k$ are 1 and $k$ itself.
- an integer $k>1$ is composite if it is not prime.
- an integer $k$ is a perfect square if and only if there exists an integer $r$ such that $k=r^{2}$


## What about...

- claim: given any non-negative integer $n$, the sum of integers up to $n$ is $n *(n+1) / 2$
- techniques we know:
- direct proofs
- proof of the contrapositive
- proof by example / disproof by counterexample
- proof by contradiction
- using cases


## (on summation notation)

- claim: given any non-negative integer $n$, the sum of the integers from 1 up to $n$ is $n^{*}(n+1) / 2$
- could also write using summation notation:


## What about . . .

- observations:
- want to prove something is true for all elements of a set (the nonnegative integers)
- the set is ordered in the sense that we can talk about the smallest/first element, then the next one, then the next one, ... (0, 1, 2, 3, ...)


## Proofs by induction

## Definition 5.1: Proof by mathematical induction.

Suppose that we want to prove that $P(n)$ holds for all $n \in \mathbb{Z} \geq 0$. To give a proof by mathematical induction of $\forall n \in \mathbb{Z} \geq 0: P(n)$, we prove two things:

1 the base case: prove $P(0)$.
2 the inductive case: for every $n \geq 1$, prove $P(n-1) \Rightarrow P(n)$.

## Structure of a proof by induction

- claim: for all x, P(x)
- we prove the claim using a proof by induction on: $x$
- base case: $\mathrm{P}\left(\mathrm{x}^{*}\right)$ holds for the smallest $\mathrm{x}^{*}$
- inductive step: $P\left(x^{\prime}\right) \rightarrow P(x)$

El If we assume $P\left(x^{\prime}\right)$ for some $x^{\prime}$ (inductive hypothesis)
E We must show that for every way we can grow $x^{\prime}$ into some $x, P\left(x^{\prime}\right) \rightarrow P(x)$

- therefore by the principle of mathematical induction: for all $x$, $P(x)$


## Structure of a proof by induction

- claim: for all natural numbers $n, P(n)$
- we prove the claim using a proof by induction on: $n$
- base case: P(0)
- inductive step: $\mathrm{P}(\mathrm{n}) \rightarrow \mathrm{P}(\mathrm{n}+1)$

El If we assume $\mathrm{P}(\mathrm{n})$ for some n (inductive hypothesis)
$\theta$ We must show that for $P(n) \rightarrow P(n+1)$

- therefore by the principle of mathematical induction: for all $n$, $P(n)$
- We will never miss a natural number with this induction scheme


## Notes on writing proofs by induction

- we prove the claim using a proof by induction <...>
- unless it's a direct proof should state the proof technique.
- base case
- show true on the smallest element of the set
- inductive hypothesis (IHOP)
- assume true for some value
- inductive step
- wts: if IHOP is on $n$, then prove for $n+1$. if IHOP is on $n-1$, then prove for $n$.
- some step in this part must refer back to the IHOP. otherwise it's definitely not a proof by induction (and may not be a proof at all)
- therefore by the principle of mathematical induction <...>
- have a concluding line


## Practice

For every positive integer, $\mathrm{n}+1<=\mathrm{n}$ * 2

- we prove the claim using a proof by induction on n :
- base case: $1+1<=1$ *2
- inductive step:

E inductive hypothesis (IH): n' $+1<=\mathrm{n}^{\prime} * 2$
el Wts: $\left(n^{\prime}+1\right)+1<=\left(n^{\prime}+1\right) * 2$
日 $\left(n^{\prime}+1\right)+1<=2$ * $n^{\prime}+2$ * 1
E We know n' $+1<=2{ }^{*} n^{\prime}$ by the IH, so it suffices to show that $1<=2$

- therefore by the principle of mathematical induction:
$\Rightarrow$ For all positive integers $n, n+1<=n * 2$


## Practice

For every list and function, map fI has the same length as I

- we prove the claim using a proof by induction on I:
- base case: map f [] has the same length as [] (by the base cases of map and length)
- inductive step:
$日_{\text {E }}$ inductive hypothesis (IH): length (map fI') = length $I^{\prime}$
E Wts: length (map $\left.f\left(x: l^{\prime}\right)\right)=$ length (x:l')
E map $f\left(x: l^{\prime}\right)=(f x):\left(m a p f l^{\prime}\right) \quad$ (second case of map)
Elength (x:l') = $1+$ length $I^{\prime}$ (second case of length)

E So we have: $1+$ length $I^{\prime}=1+$ length $I^{\prime}$
( Therefore, by induction, forall f and I , length (map f I$)=$ length I


## Practice

For every positive integer $n$, the sum from 1 up to $n$ is equal to $\mathrm{n} *(\mathrm{n}+1) / 2$.

- we prove the claim using a proof by induction on n :
- base case: for $\mathrm{n}=1, \ldots$
- inductive step: (for all $n^{\prime}$, if $P\left(n^{\prime}\right)$ then $P\left(n^{\prime}+1\right)$ )

E inductive hypothesis (IH): for $n=n^{\prime} \ldots$
e Wts: for $\mathrm{n}=\mathrm{n}^{\prime}+1, \ldots$
B ...

- therefore by the principle of mathematical induction:
- For all positive integers $n$, the sum from 1 up to $n$ is

$$
n *(n+1) / 2
$$

## Practice

- Identify the smallest positive integer p such that for all n $>=\mathrm{p}, \mathrm{n}!>2^{\mathrm{n}}$
- Prove that your choice of $p$ is correct
- statement needs to be true for all n

$$
\begin{aligned}
& n!= \\
& 0!=1!=1 \\
& n!\text { is read "n } \\
& \text { factorial" }
\end{aligned}
$$

- if $p>1$, need to show that the statement is not true for $p-1$

