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Mathematical Foundations

Strong induction

CS51 – Spring 2026

Proof by induction

- From last time:
  - To prove that a property $P(n)$ holds for all non-negative integers n, we prove that:
    - $P(0)$ is true,
    - for every $\mathrm{n} \geq 0$, if $P(n)$ is true, then $P(n+1)$ is true, too.
  - The proof of $P(0)$ is called the **base case**.
  - The proof that $P(n) \Rightarrow P(n+1)$ is called the **inductive case**.
  - Practically, the principle of mathematical induction says the following: to prove that a statement $P(n)$ is true for all non-negative integers n, we can prove that P “starts being true” (the base case) and that P “never stops being true” (the inductive case).

Proof by strong induction

- To prove that a property $P(n)$ holds for all non-negative integers n, we prove that:
  - $P(0)$ is true,
  - for every $n \geq 0$, if $P(0) \land P(1) \land \cdots \land P(n)$ is true, then $P(n+1)$ is true, too.
- The proof of $P(0)$ is called the **base case**.
- The proof that $P(0) \land P(1) \land \cdots \land P(n) \Rightarrow P(n+1)$ is called the **inductive case**.
- Practically, what we saw last time is known as **weak induction**. In weak induction: we show that if P is true this time, then it will be true next time. We only assume $P(n)$.
- In **strong induction**, we show that if P has been true up until now, then it will be true next time. We assume all of previous steps $P(0) \land P(1) \land \cdots \land P(n)$ to show $P(n+1)$.
- Stronger doesn't mean better, just that we have a stronger inductive hypothesis.

An aside: floor and ceiling

- The **floor** of a number x, written as $\lfloor x \rfloor$ is the largest integer that is less than or equal to x.
  - For example, $\lfloor 4.7 \rfloor = 4$.
- The **ceiling** of a number x, written as $\lceil x \rceil$ is the smallest integer that is greater than or equal to x.
  - For example, $\lceil 4.7 \rceil = 5$.

A reminder: moduli and division

- For any integers $\mathrm{k} > 0$ and n, the integer $n \bmod \mathrm{k}$ (also denoted as $n \,\%\, \mathrm{k}$) is the remainder when we divide n by k. The value of $n \bmod \mathrm{k}$ is $n - k\left\lfloor \frac{n}{k} \right\rfloor$.
  - For example, $8\ mod\ 3 = 2$ because $8 = 3 \times 2 + 2$ or by definition $8 - 3\left\lfloor \frac{8}{3} \right\rfloor = 8 - 3 \times 2 = 2$.
- For any integer $\mathrm{k} > 0$ and n, we say that k evenly divides n (or n is evenly divisible by k) , written $k|n$, if $\frac{n}{k}$ is an integer. Note that $k|n$ is equivalent to $n \bmod \mathrm{k} = 0$.
  - For example, $5|10$ but $5 \nmid 9$.
- A positive integer $n \;\geq\; 1$ is **prime** if the only positive integers that evenly divide n are 1 and n itself. If a number $n > 1$ is not prime it is called **composite**.

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- 3. Inductive case ($n \geq 2$): Assume the inductive hypothesis $P(2) \wedge P(3) \wedge \cdots \wedge P(n)$, that is assume for any integer $n \geq 2$, we have that $2, 3, 4, \ldots, n$ are evenly divisible by a prime number.

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- We must prove $P(n+1)$, that is we must prove $n + 1$ is also evenly divisible by a prime number.

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  - By the inductive hypothesis, a is evenly divisible by a prime number p, therefore $n + 1$ is also divisible by p.

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- If $n + 1$ is not prime, then there exists an integer $a,\ \ 2 \leq a \leq n$ so that $a|(n + 1)$.
  - By the inductive hypothesis, a is evenly divisible by a prime number p, therefore $n + 1$ is also divisible by p.
- Hence, regardless of whether $n + 1$ is prime or not, $n + 1$ is divisible by a prime number and we have proven $P(n + 1)$. By the principle of mathematical induction, we have proven our claim $P(n)$ holds for any integer $n \geq 2$.

Practice time

- Define a sequence of numbers $a_1, a_2, a_3, \ldots$ as follows:
- $a_1 = 0$
- $a_2 = 2$
- $a_n = 3 \times a_{\lfloor \frac{n}{2} \rfloor} + 2$, for all integers $\mathrm{n} \geq 3$.
- 1) Calculate $a_3, a_4, a_{5,} a_6, a_7$.
- 2) Let $P(n)$ be the property "a_n is even." Use strong mathematical induction to show that this property holds for all integers $n \geq 1$.

Answer – Part 1

- $a_1 = 0$
- $a_2 = 2$
- $a_3 = 3 \times a_{\left\lfloor \frac{3}{2} \right\rfloor} + 2 = 3 \times a_1 + 2 = 3 \times 0 + 2 = 2$
- $a_4 = 3 \times a_{\left\lfloor \frac{4}{2} \right\rfloor} + 2 = 3 \times a_2 + 2 = 3 \times 2 + 2 = 8$
- $a_5 = 3 \times a_{\left\lfloor \frac{5}{2} \right\rfloor} + 2 = 3 \times a_2 + 2 = 3 \times 2 + 2 = 8$
- $a_6 = 3 \times a_{\left\lfloor \frac{6}{2} \right\rfloor} + 2 = 3 \times a_3 + 2 = 3 \times 2 + 2 = 8$
- $a_7 = 3 \times a_{\left\lfloor \frac{7}{2} \right\rfloor} + 2 = 3 \times a_3 + 2 = 3 \times 2 + 2 = 8$

Answer – Part 2

- 1. Using strong induction, we will prove the property $P(n)$ that a_n is even for all integers $\mathrm{n} \geq 1$.
- 2. Base cases ($n = 1, n = 2$): The property holds because $a_1 = 0$ and $a_2 = 2$.

Answer – Part 2

- 3. Inductive case ($n \geq 1$): Assume the inductive hypothesis $P(1) \wedge P(2) \wedge \cdots \wedge P(n)$, that is assume for every integer $1, 2, 3, \ldots, n, n > 2$, we have that $a_1, a_2, \ldots, a_n$ are even
- We must prove $P(n+1)$, that is we must prove that a_{n+1} is even.
- By definition $a_{n+1} = 3 \times a_{\left\lfloor \frac{n+1}{2} \right\rfloor} + 2$ for all integers greater or equal to 3.
- $a_{\left\lfloor \frac{n+1}{2} \right\rfloor}$ is even by induction hypothesis, because $n + 1 > 2$ and and so $0 < \left\lfloor \frac{n+1}{2} \right\rfloor < n + 1$.
- Thus, $3 \times a_{\left\lfloor \frac{n+1}{2} \right\rfloor}$ is even (because odd x even = even).
- Hence, $3 \times a_{\left\lfloor \frac{n+1}{2} \right\rfloor} + 2$ (because even + even=even). Consequently, a_{n+1} is even!

Weak vs strong induction

- Anything that can be proven using weak induction can also be shown using strong induction.
- However, if you can prove something using weak induction, you should.