Advanced Algorithms

October 16, 2025

Logistics

- Welcome back
- Reminder: assignment due next Tuesday, late days available
- Office hours today after class
- Course feedback:
    - Exercise sets
    - Length of assignment
    - Assign groups
    - Slides / notes
- You can always submit more feedback, see course webpage

Hot Tub LP Revisited

- You are running a Hot Tub production company.
- You can produce two types of hot tubs: Aqua-Spas and Hydro-Luxes.
- They require resources (pumps, labor, and tubing), and yield a certain profit

	Aqua-Spa	Hydro-Lux
Pumps	1	1
Labor	9 hours	6 hours
Tubing	12 feet	16 feet
Price	$350	$300

- You have 200 pumps, 1566 hours of labor, and 2880 feet of tubing.
- How many of each hot tub to produce if we want to maximize sales?

Linear Programming relaxation:

x_A = number of Aqua-Spas to produce

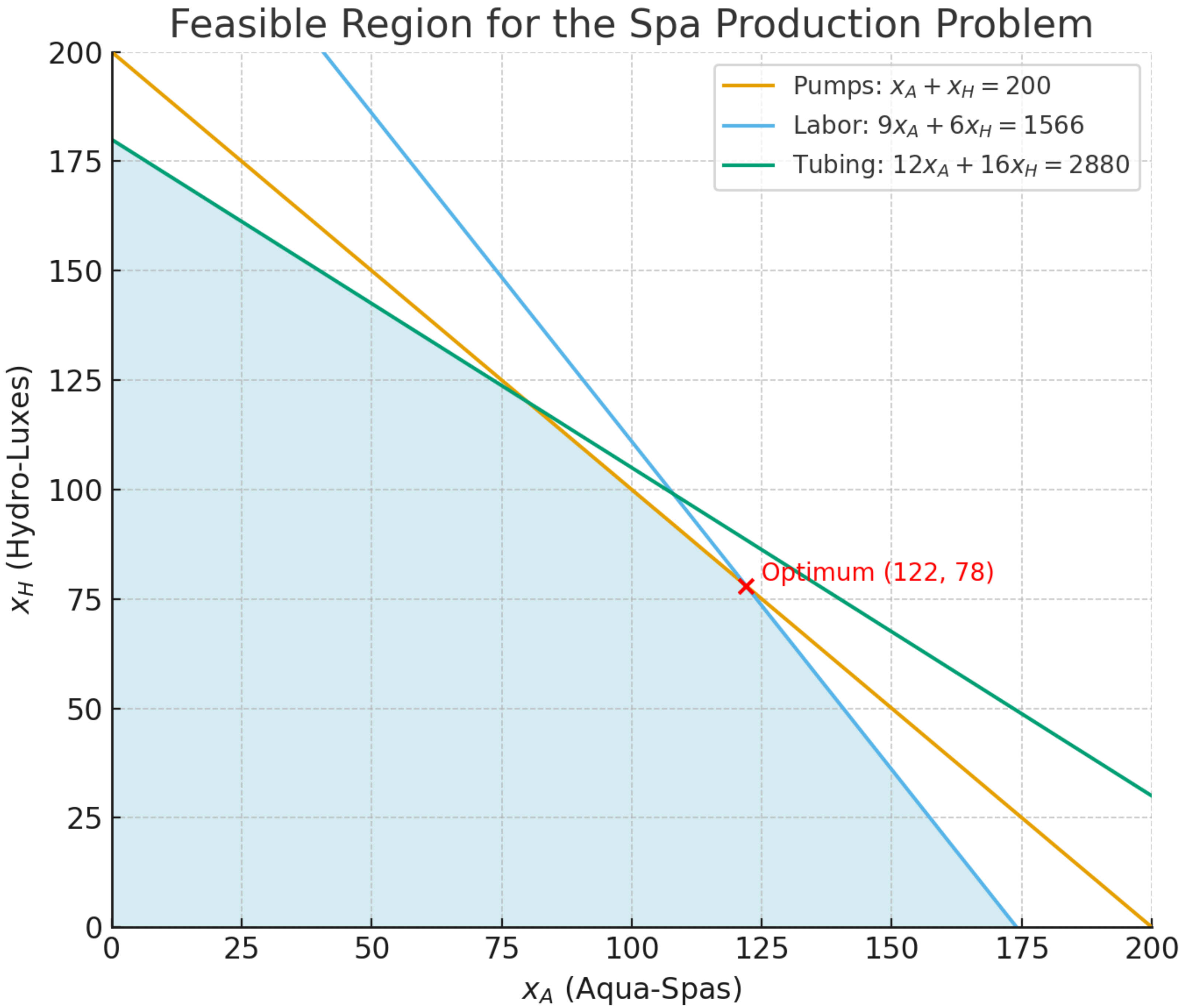
x_H = number of Hydro-Luxes to produce

Maximize: $350x_A + 300x_H$

Subject to:

$$\begin{array}{ll} x_A + x_H \leq 200 & \text{(pumps)} \\ 9x_A + 6x_H \leq 1566 & \text{(labor)} \\ 12x_A + 16x_H \leq 2880 & \text{(tubing)} \\ x_A, x_H \geq 0 & \text{(non-negativity)} \end{array}$$

Optimal value: \$66,100


Feasible Region for the Spa Production Problem
Pumps: $x_A + x_H = 200$
Labor: $9x_A + 6x_H = 1566$
Tubing: $12x_A + 16x_H = 2880$
Optimum (122, 78)
x_H (Hydro-Luxes)
x_A (Aqua-Spas)
0
25
50
75
100
125
150
175
200

Goal: upper bound the profit

Maximize: $350x_A + 300x_H$

Subject to:

$$x_A + x_H \leq 200 \quad \text{(pumps)}$$
$$9x_A + 6x_H \leq 1566 \quad \text{(labor)}$$
$$12x_A + 16x_H \leq 2880 \quad \text{(tubing)}$$
$$x_A, x_H \geq 0 \quad \text{(non-negativity)}$$

Upper Bounds

- Let's call the optimal value z^*

- Start with the pumps constraint :
  - $x_A + x_H \leq 200$

- Multiply both sides by 350:
  - $350x_A + 350x_H \leq 70{,}000$

- Observe upper bound:
  - $z^* \leq 70{,}000$

- Note: we relied on the non-negativity constraints for this conclusion

Upper Bounds

- Try another constraint now

- Start with the labor constraint :
  - $9x_A + 6x_H \leq 1566$

- Multiply both sides by 50:
  - $450x_A + 300x_H \leq 78{,}300$

- Observe upper bound:
  - $z^* \leq 78{,}300$

- Note: we relied on the non-negativity constraints for this conclusion

Combining constraints

- Start with the pumps and labor constraints:
  - $x_A + x_H \leq 200$
  - $9x_A + 6x_H \leq 1566$

- Multiply pumps constraint by 300, labor constraint by 6:
  - $300x_A + 300x_H \leq 60{,}000$
  - $54x_A + 36x_H \leq 9{,}396$

- Add together
  - $354x_A + 336x_H \leq 69{,}396$

- Observe upper bound:
  - $z^* \leq$ 69,396

General Strategy

- Start with the constraints:
  - $x_A \; + x_H \;\; \leq \; 200$
  - $9x_A + 6x_H \;\; \leq \; 1566$
  - $12x_A \; + 16x_H \;\; \leq \; 2880$
- Choose any three non-negative values y_1, y_2, y_3 and multiply:
  - $y_1(x_A \; + \; x_H) \; \leq y_1(200)$
  - $y_2(9x_A \; + 6x_H) \leq y_2(1566)$
  - $y_3(12x_A \; + 16x_H) \leq y_3(2880)$
- Check that the objective is “beaten”
  - $y_1 + 9y_2 + 12y_3 \geq 350$
  - $y_1 + 6y_2 + 16y_3 \geq 300$
- Observe upper bound:
  - $z^* \leq \; 200y_1 \; + \; 1566y_2 \; + \; 2880y_3$

General Strategy

- What's the best (lowest) upper bound we can get using this strategy?
- BIG idea: use a Linear Program.
- Decisions: y_1, y_2, y_3
- Objective: $\min\ 200y_1 + 1566y_2 + 2880y_3$
- Constraints:
  - $y_1 + 9y_2 + 12y_3 \geq 350$
  - $y_1 + 6y_2 + 16y_3 \geq 300$
  - $y_1, y_2, y_3 \geq 0$

This LP is the dual of the original LP

Duality

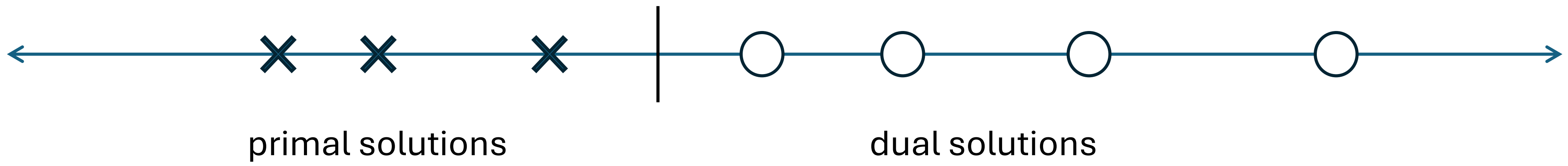
Primal:

$$
\begin{array}{ll}
\max & 350x_{\mathrm{A}} + 300x_{H} \\
s.t. & x_{A} + \mathrm{x}_{H} \leq 200 \\
 & 9x_{A} + 6\mathrm{x}_{H} \leq 1566 \\
 & 12x_{A} + 16x_{H} \leq 2880 \\
 & x_{\mathrm{A}}, \mathrm{x}_{H} \geq 0
\end{array}
$$

Dual:

$$
\begin{array}{ll}
\min & 200\mathrm{y}_{1} + 1566\mathrm{y}_{2} + 2880y_{3} \\
s.t. & \mathrm{y}_{1} + 9\mathrm{y}_{2} + 12\mathrm{y}_{3} \geq 350 \\
 & \mathrm{y}_{1} + 6\mathrm{y}_{2} + 16\mathrm{y}_{3} \geq 300 \\
 & \mathrm{y}_{1}, \mathrm{y}_{2}, \mathrm{y}_{3} \geq 0
\end{array}
$$

By construction, every feasible solution to the dual yields an upper bound on the value of the primal



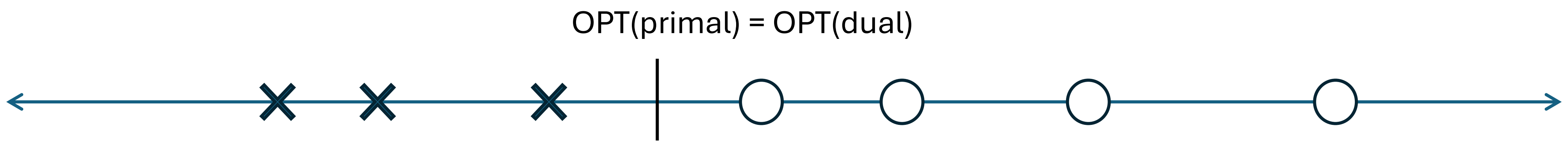
Duality

All Linear Programs come in dual pairs. One maximizes one minimizes.

Weak Duality: given a primal LP and its dual LP:

objective value of any feasible solution to the maximization problem	$\leq$	objective value of any feasible solution to the minimization problem

Strong Duality: if primal has an optimal solution, then so does its dual, and their optimal values are equal.



Your turn:

Consider the following LP:

$$\max 2x_1 + 7x_2 + 4x_3$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 10 \\ 3x_1 + 3x_2 + 2x_3 &\leq 10 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Show that the optimal value cannot exceed 25

Duality in Algorithm Design