Mergeable Heaps

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CS140
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Admin

Assignment 2 graded

Assignment 3

Assignment 4

2

Another set data structure

Idea: store data in a collection of arrays

- array *i* has size 2^i
- an array is either full or empty (never partially full)
- each array is stored in sorted order
- no relationship between arrays

Binary array set

https://www.nayuki.io/page/binary-array-set#:~:text=The%20binary%20array%20set%20data,a%20BAS%20is%20practically%20negligible.

3

Binary heap

A binary tree where the value of a parent is greater than or equal to the value of its children

Additional restriction: all levels of the tree are **complete** except the last

Max heap vs. min heap

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Binary heap - operations

Max - return the largest element in the set

ExtractMax – Return and remove the largest element in the set

Insert(val) – insert val into the set

IncreaseElement(x, val) – increase the value of element x to val

BuildHeap(A) – build a heap from an array of elements

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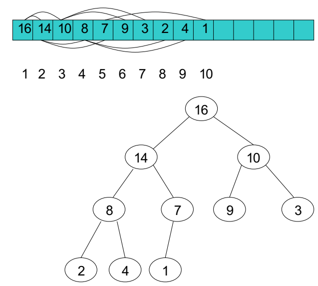
Binary heap representations



6

Heapify

Assume left and right children are heaps, turn current node into a valid heap

HEAPIFY(A, i) aka "sink"

```
l ← LEFT(i)
r ← RIGHT(i)
largest ← i
if l ≤ heap-size[A] and A[l] > A[i]
        largest ← l
if r ≤ heap-size[A] and A[r] > A[largest]
        largest ← r
if largest ≠ i
        swap A[i] and A[largest]
        HEAPIFY(A, largest)
```

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```

find out which is largest: current, left of right

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Heapify

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```

if a child is larger, swap and recurse

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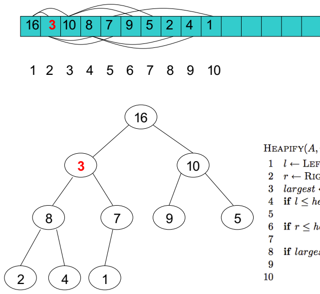
Heapify

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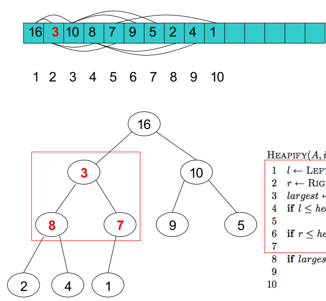
Heapify

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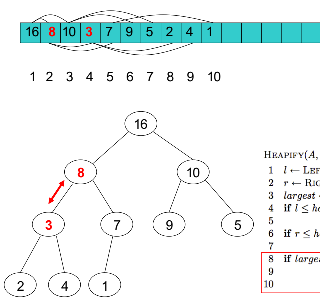
Heapify

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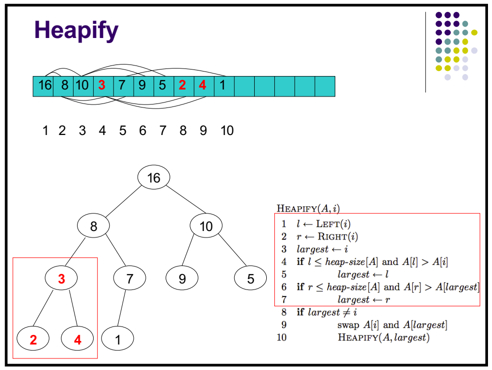
Heapify
16 8 10 3 7 9 5 2 4 1
1 2 3 4 5 6 7 8 9 10
16
8 10
3 7 9 5
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HEAPIFY(A, i)
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6 if r ≤ heap-size[A] and A[r] > A[largest]
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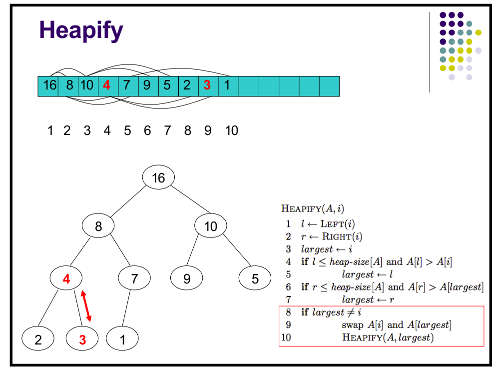
8 if largest ≠ i
9 swap A[i] and A[largest]
10 HEAPIFY(A, largest)

13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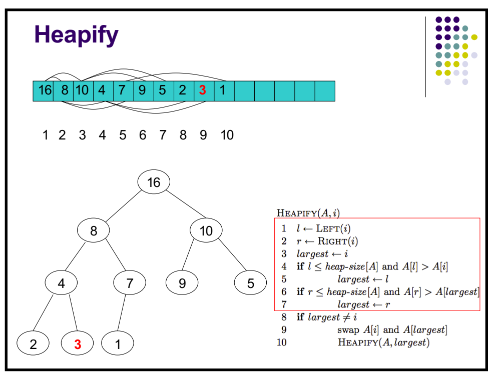
14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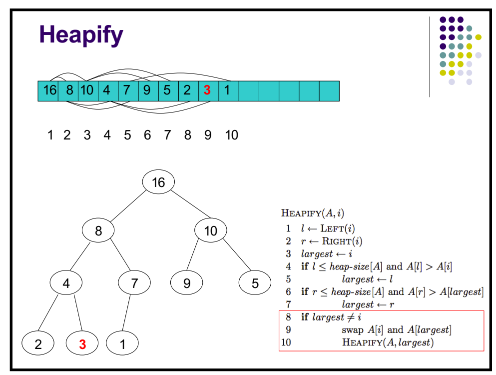
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8 if largest ≠ i
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10 HEAPIFY(A, largest)

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Running time of Heapify

O(height of the tree)

What is the height of the tree?

- Complete binary tree, except for the last level

$$2^h \le n$$

$$h \le \log_2 n$$

O(log n)

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Binary heap - operations

Max - return the largest element in the set

ExtractMax – Return and remove the largest element in the set

Insert(val) – insert val into the set

IncreaseElement(x, val) – increase the value of element x to val

BuildHeap(A) – build a heap from an array of elements

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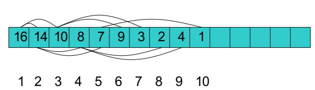
Max

What is the largest element in the set?

Return A[1]



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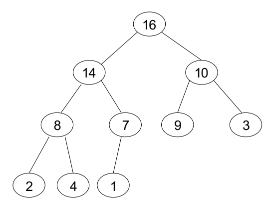
ExtractMax

Return and remove the largest element in the set



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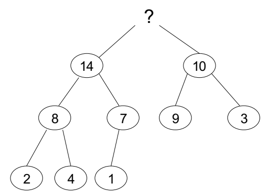
ExtractMax

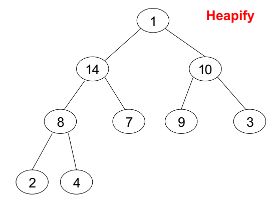
Return and remove the largest element in the set



25

ExtractMax

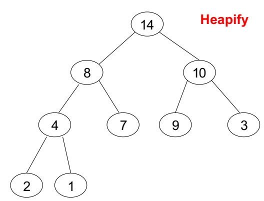
Return and remove the largest element in the set



30

ExtractMax

Return and remove the largest element in the set



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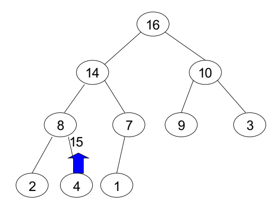
IncreaseElement (aka swim up)

Increase the value of element *x* to *val*



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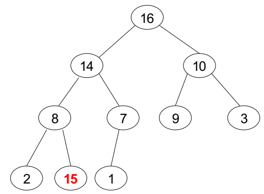
IncreaseElement

Increase the value of element *x* to *val*



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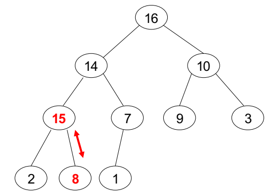
IncreaseElement

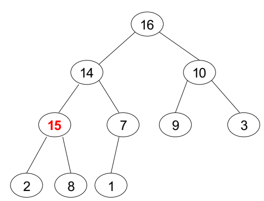
Increase the value of element *x* to *val*



36

IncreaseElement

Increase the value of element *x* to *val*



37

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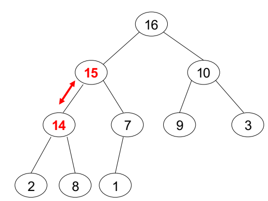
Increase the value of element *x* to *val*



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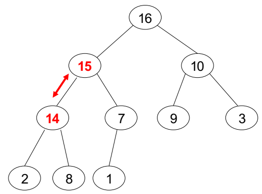

Runtime?

O(height of tree) = O(log n)

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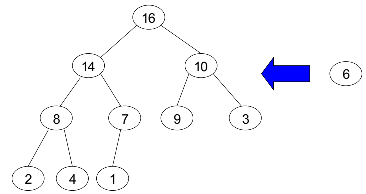
Insert

Insert *val* into the set



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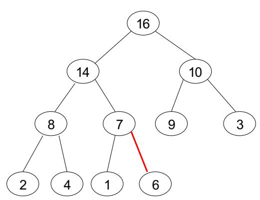

45

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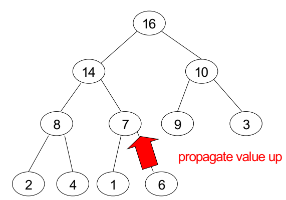
Insert *val* into the set



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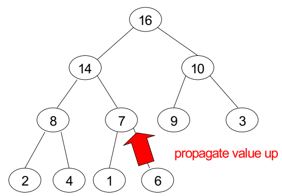
Insert

Insert *val* into the set



O(log n)

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Building a heap

Can we build a heap using the functions we have so far?

- Max
- ExtractMax
- Insert(val)
- IncreaseElement(x, val)

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Building a heap

For each element x in array:
    insert(x)

```
BUILD-HEAP1(A)
copy A to B
heap-size[A] ← 0
for i ← 1 to length[B]
        INSERT(A, B[i])
```

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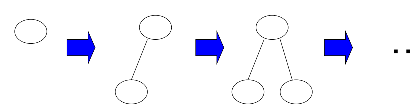
Running time of BuildHeap1

n calls to Insert – O(n log n)

Can we do better?



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Building a heap: take 2

```
BUILD-HEAP2(A)
  heap-size[A] ← (length)[A]
  for i ← ⌊(length)[A]/2⌋ to 1
        HEAPIFY(A, i)
```

Start with n/2 “one-node” heaps

call Heapify on element n/2-1, n/2-2, n/2-3 …

all children have smaller indices

building from the bottom up, makes sure that all the children are heaps

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```
BUILD-HEAP2(A)
  heap-size[A] ← (length)[A]
  for i ← ⌊(length)[A]/2⌋ to 1
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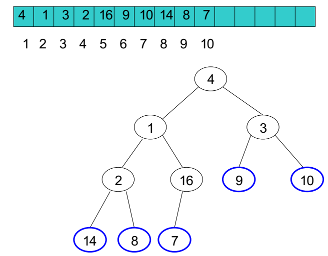

54


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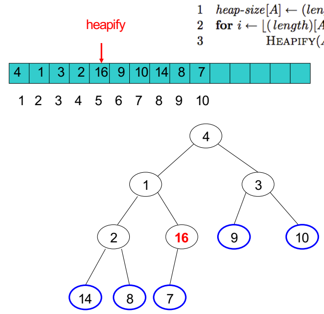

55


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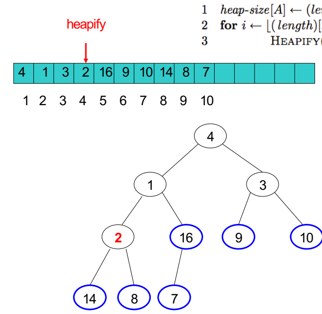
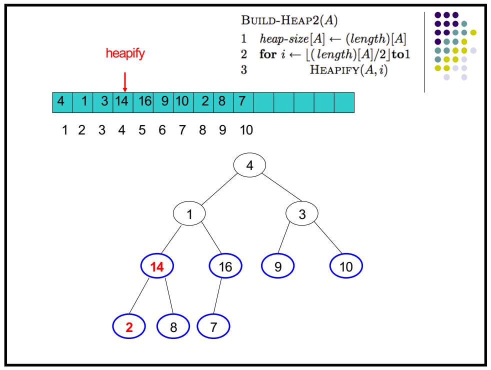


56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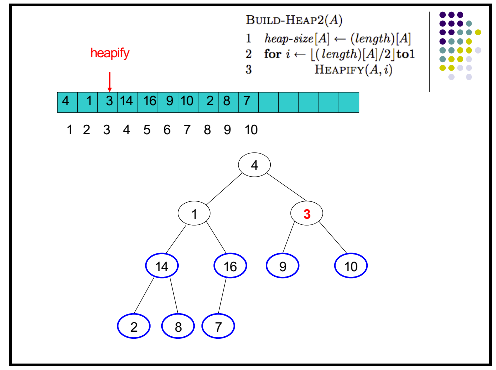
HEAPIFY(A, i)
heapify
4 1 3 14 16 9 10 2 8 7
1 2 3 4 5 6 7 8 9 10

57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heapify
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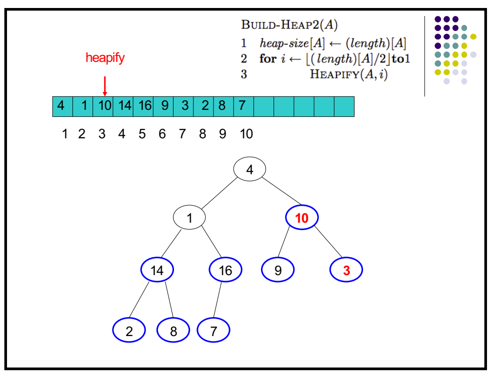
58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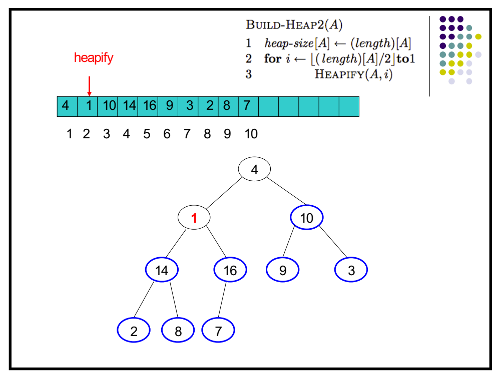
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heapify
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1 2 3 4 5 6 7 8 9 10

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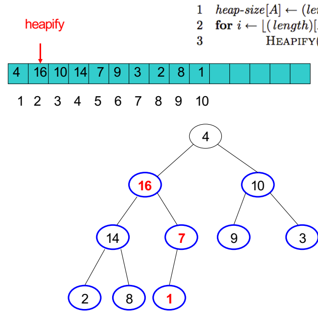
BUILD-HEAP2(A)

$heap\text{-}size[A] \leftarrow (length)[A]$

for $i \leftarrow \lfloor (length)[A]/2 \rfloor$ **to** 1

    HEAPIFY(A, i)



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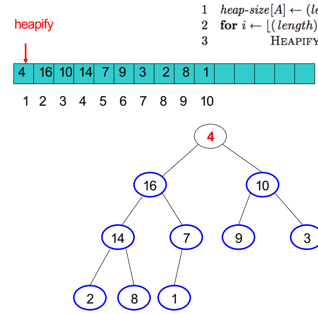

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BUILD-HEAP2(A)

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for $i \leftarrow \lfloor (length)[A]/2 \rfloor$ **to** 1

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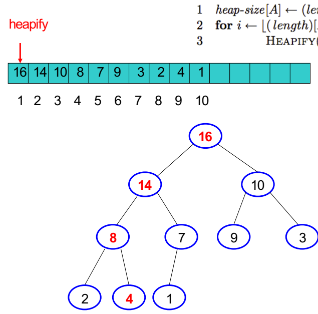

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Running time of BuildHeap2

n/2 calls to Heapify – O(n log n)

Can we get a tighter bound?

BUILD-HEAP2(A)

$heap\text{-}size[A] \leftarrow (length)[A]$

for $i \leftarrow \lfloor (length)[A]/2 \rfloor$ **to** 1

    HEAPIFY(A, i)

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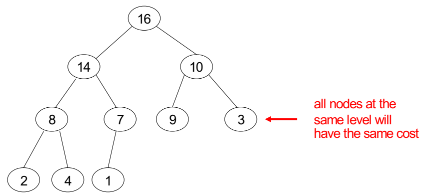
Running time of BuildHeap2



How many nodes are at level *d*? 2^d

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Running time of BuildHeap2

$$T(n) = \sum_{d=0}^{\log n} 2^d O(d)$$

?

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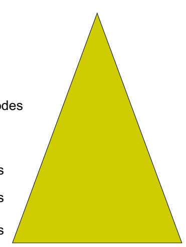
Nodes at height *h*

h < ceil(n/2^{h+1}) nodes

h=2 < ceil(n/8) nodes

h=1 < ceil(n/4) nodes

h=0 < ceil(n/2) nodes



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Running time of BuildHeap2

$$T(n) = \sum_{h=0}^{\log n} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h)$$

$$= O\left(n \sum_{h=0}^{\log n} \left\lceil \frac{1}{2^{h+1}} \right\rceil h \right)$$

$$= O\left(n \sum_{h=0}^{\log n} \frac{h}{2^h} \right)$$

$$= O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h} \right)$$

$$= \theta(n)$$

$$\sum_{h=0}^{\infty} \frac{h}{2^h} = \frac{1/2}{(1-1/2)^2} = 2$$

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Binary heaps

Procedure	Binary heap (worst-case)
BUILD-HEAP	$\Theta(n)$
INSERT	$\Theta(\log n)$
MAXIMUM	$\Theta(1)$
EXTRAC-MAX	$\Theta(\log n)$
UNION	
INCREASE-ELEMENT	$\Theta(\log n)$
DELETE	$\Theta(\log n)$

(adapted from Figure 19.1, pg. 456 [1])

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Mergeable heaps

Procedure	Binary heap (worst-case)
BUILD-HEAP	$\Theta(n)$
INSERT	$\Theta(\log n)$
MAXIMUM	$\Theta(1)$
EXTRAC-MAX	$\Theta(\log n)$
UNION	
INCREASE-ELEMENT	$\Theta(\log n)$
DELETE	$\Theta(\log n)$

(adapted from Figure 19.1, pg. 456 [1])

- Mergeable heaps support the union operation
- Allows us to combine two heaps to get a single heap
- Union runtime for binary heaps?

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Union for binary heaps

Procedure	Binary heap (worst-case)
BUILD-HEAP	$\Theta(n)$
INSERT	$\Theta(\log n)$
MAXIMUM	$\Theta(1)$
EXTRAC-MAX	$\Theta(\log n)$
UNION	$\Theta(n)$
INCREASE-ELEMENT	$\Theta(\log n)$
DELETE	$\Theta(\log n)$

(adapted from Figure 19.1, pg. 456 [1])

concatenate the arrays and then call Build-Heap

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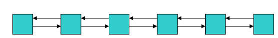
Linked-list heap

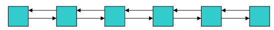


Store the elements in an unordered doubly linked list

- Insert:
- Max:
- Extract-Max:
- Increase:
- Union:

77

Linked-list heap



Store the elements in an unordered doubly linked list

- Insert: add to the end/beginning
- Max: search through the linked list
- Extract-Max: search and delete
- Increase: increase value
- Union: concatenate linked lists

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Linked-list heap

Procedure	Binary heap (worst-case)	Linked-list
BUILD-HEAP	$\Theta(n)$	$\Theta(n)$
INSERT	$\Theta(\log n)$	$\Theta(1)$
MAXIMUM	$\Theta(1)$	$\Theta(n)$
EXTRAC-MAX	$\Theta(\log n)$	$\Theta(n)$
UNION	$\Theta(n)$	$\Theta(1)$
INCREASE-ELEMENT	$\Theta(\log n)$	$\Theta(1)$
DELETE	$\Theta(\log n)$	$\Theta(1)$

(adapted from Figure 19.1, pg. 456 [1])

Faster Union, Increase, Insert and Delete… but slower Max operations

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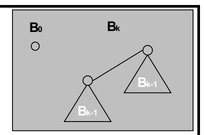
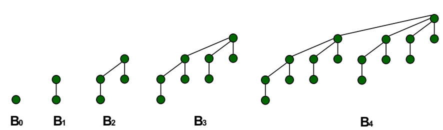
Adapted from:
Kevin Wayne

Binomial Tree



B_k: a binomial tree B_{k-1} with the addition of a left child with another binomial tree B_{k-1}



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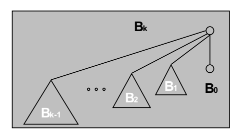
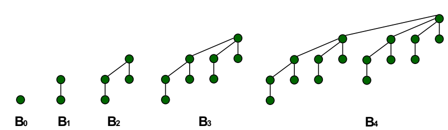
Binomial Tree

Number of nodes with respect to k?



$N(B_o) = 1$
$N(B_k) = 2\ N(B_{k-1}) = 2^k$



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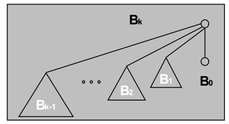
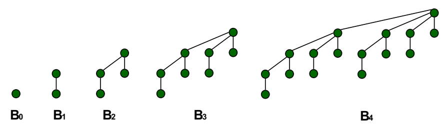
Binomial Tree

Height?



$H(B_0) = 1$
$H(B_k) = 1 + H(B_{k-1}) = k$



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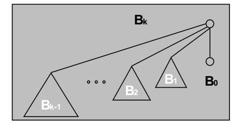
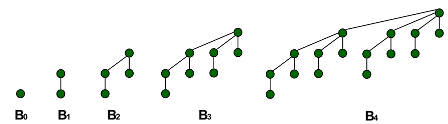
Binomial Tree

Degree of root node?



k, each time we add another binomial tree



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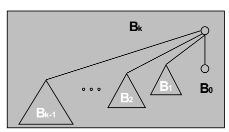
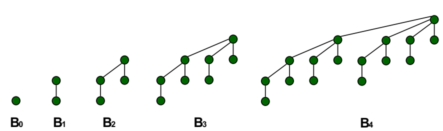
Binomial Tree

What are the children of the root?



k binomial trees:
$B_{k-1}, B_{k-2}, \ldots, B_0$



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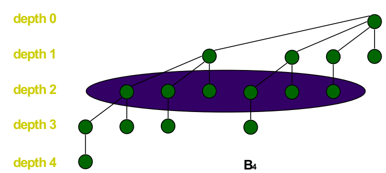
Binomial Tree

Why is it called a binomial tree?



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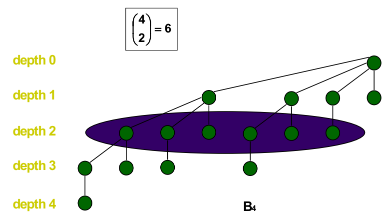
Binomial Tree

B_k has $\binom{k}{i}$ nodes at depth i.



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Another set data structure: recap

Idea: store data in a collection of arrays

- array *i* has size 2^i
- an array is either full or empty (never partially full)
- each array is stored in sorted order
- no relationship between arrays

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Another set data structure: recap

Which arrays are full and empty are based on the number of elements

- specifically, binary representation of the number of elements
- 4 items = 100 = A2-full, A1-empty, A_0-empty
- 11 items = 1011 = A_3-full, A_2-empty, A_1-full, A_0-full

A_0: [5]
A_1: [4, 8]
A_2: empty
A_3: [2, 6, 9, 12, 13, 16, 20, 25]

Lookup: binary search through each array

- Worst case runtime?

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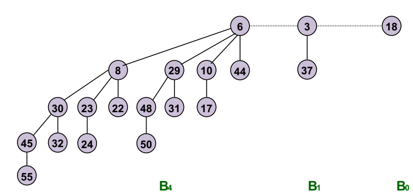
Binomial Heap

Binomial heap Vuillemin, 1978.

Sequence of binomial trees that satisfy binomial heap property:

- each tree is min-heap ordered
- top level: full or empty binomial tree of order k
- which are empty or full is based on the number of elements



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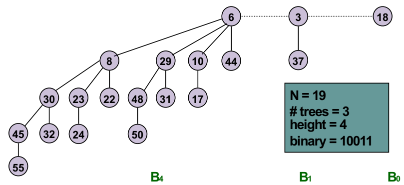
Binomial Heap

A_0: [18]
A_1: [3, 7]
A_2: empty
A_3: empty
A_4: [6, 8, 29, 10, 44, 30, 23, 22, 48, 31, 17, 45, 32, 24, 55]



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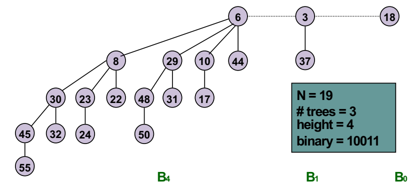
Binomial Heap: Properties

How many heaps?

O(log n) – binary number representation



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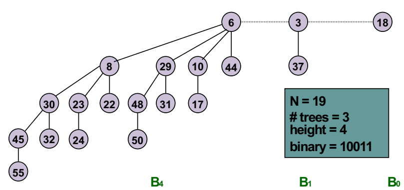
Binomial Heap: Properties

Where is the max/min?

Must be one of the
roots of the heaps



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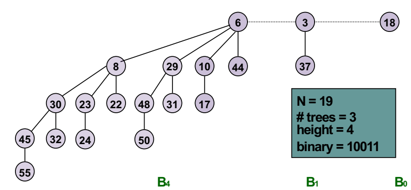
Binomial Heap: Properties

Runtime of max/min?

O(log n)



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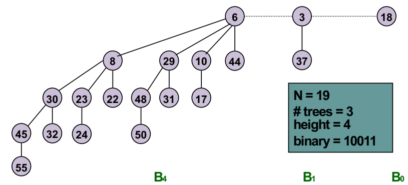
Binomial Heap: Properties

Height?

$\log_2 n$

- largest tree = $B_{\log n}$
- height of that tree is log n



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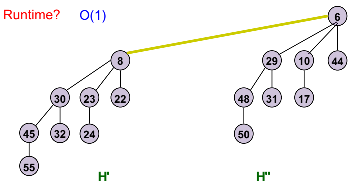
Binomial Heap: Union

How can we merge two binomial tree heaps of the same size (2^k)?

- connect roots of H' and H''
- choose smaller key to be root of H

Runtime? O(1)



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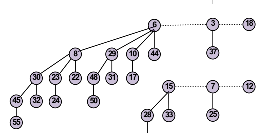
Binomial Heap: Union



How can we combine/merge binomial heaps (i.e. a combination of binomial tree heaps)?

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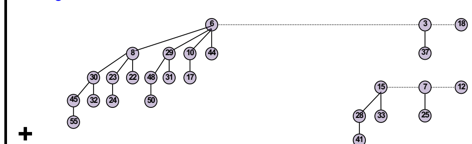
Binomial Heap: Union

Go through each tree size starting at 0 and merge as we go



19 + 7 = 26

		1	1	1	
	1	0	0	1	1
+	0	0	1	1	1
	1	1	0	1	0

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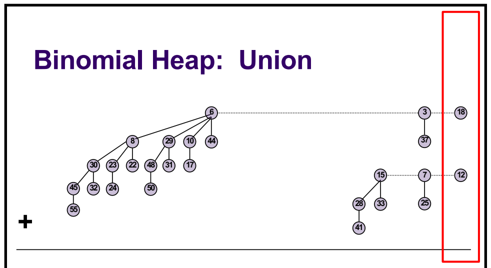
Binomial Heap: Union



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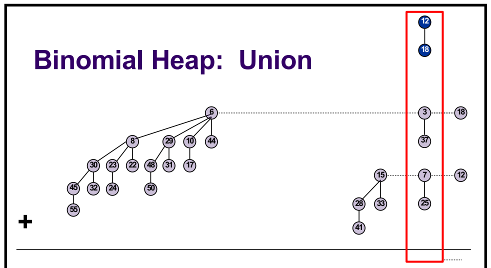
Binomial Heap: Union



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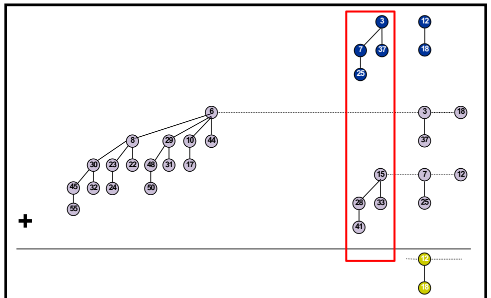
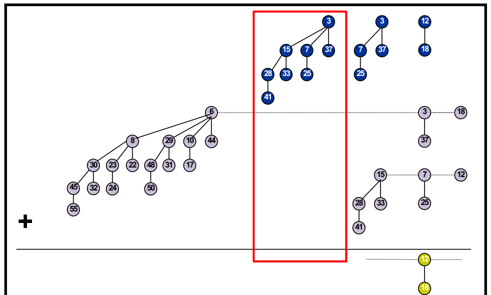
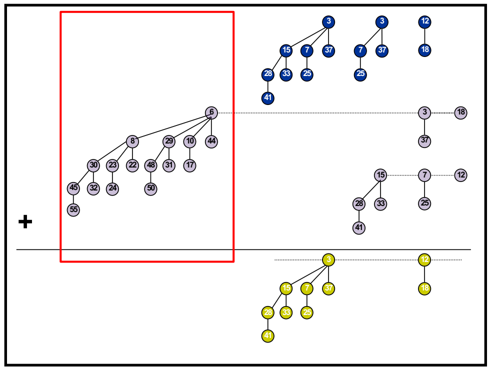


100



101



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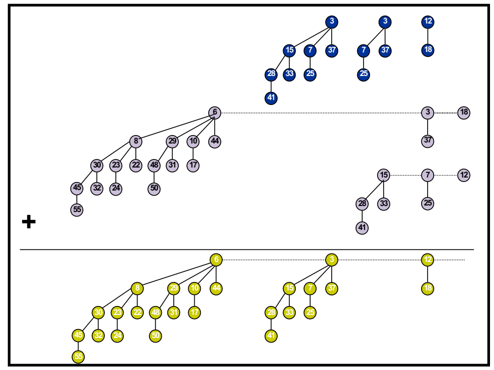

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Binomial Heap: Union

Analogous to binary addition

Running time?

- Proportional to number of trees in root lists 2 O($\log_2$ N)
- O(log N)

19 + 7 = 26

		1	1	1	
	1	0	0	1	1
+	0	0	1	1	1
	1	1	0	1	0

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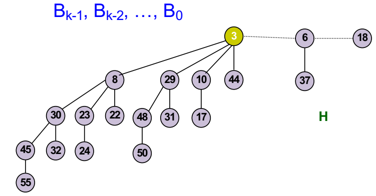
Binomial Heap: Delete Min/Max

We can find the min/max in O(log n).
How can we extract it?

Hint: B_k consists of binomial trees:
B_{k-1}, B_{k-2}, …, B_0



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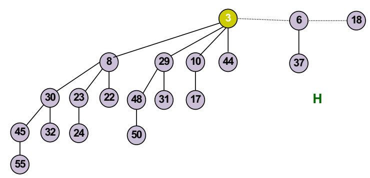
Binomial Heap: Delete Min

Delete node with minimum key in binomial heap H.

- Find root x with min key in root list of H, and delete
- H' ← broken binomial trees
- H ← Union(H', H)



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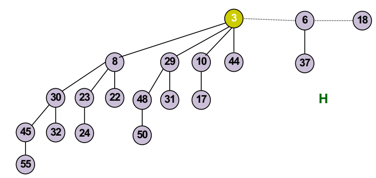
Binomial Heap: Delete Min

Delete node with minimum key in binomial heap H.

- Find root x with min key in root list of H, and delete
- H' ← broken binomial trees
- H ← Union(H', H)

Running time? O(log N)



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Heaps

Procedure	Binary heap (worst-case)	Binomial heap (worst-case)
BUILD-HEAP	$\Theta(n)$	$\Theta(n)$
INSERT	$\Theta(\log n)$	$O(\log n)$
MAXIMUM	$\Theta(1)$	$O(\log n)$
EXTRAC-MAX	$\Theta(\log n)$	$\Theta(\log n)$
UNION	$\Theta(n)$	$\Theta(\log n)$
INCREASE-ELEMENT	$\Theta(\log n)$	$\Theta(\log n)$
DELETE	$\Theta(\log n)$	$\Theta(\log n)$

(adapted from Figure 19.1, pg. 456 [1])

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Fibonacci Heaps

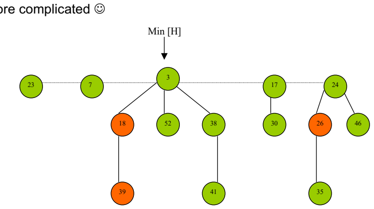
Similar to binomial heap

- A Fibonacci heap consists of a sequence of heaps

More flexible

- Heaps do not have to be binomial trees

More complicated ☺



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Heaps

Procedure	Binary heap (worst-case)	Binomial heap (worst-case)	Fibonacci heap (amortized)
BUILD-HEAP	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
INSERT	$\Theta(\log n)$	$O(\log n)$	$\Theta(1)$
MAXIMUM	$\Theta(1)$	$O(\log n)$	$\Theta(1)$
EXTRAC-MAX	$\Theta(\log n)$	$\Theta(\log n)$	$O(\log n)$
UNION	$\Theta(n)$	$\Theta(\log n)$	$\Theta(1)$
INCREASE-ELEMENT	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(1)$
DELETE	$\Theta(\log n)$	$\Theta(\log n)$	$O(\log n)$

(adapted from Figure 19.1, pg. 456 [1])

Should you always use a Fibonacci heap?

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Heaps

Procedure	Binary heap (worst-case)	Binomial heap (worst-case)	Fibonacci heap (amortized)
BUILD-HEAP	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
INSERT	$\Theta(\log n)$	$O(\log n)$	$\Theta(1)$
MAXIMUM	$\Theta(1)$	$O(\log n)$	$\Theta(1)$
EXTRAC-MAX	$\Theta(\log n)$	$\Theta(\log n)$	$O(\log n)$
UNION	$\Theta(n)$	$\Theta(\log n)$	$\Theta(1)$
INCREASE-ELEMENT	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(1)$
DELETE	$\Theta(\log n)$	$\Theta(\log n)$	$O(\log n)$

(adapted from Figure 19.1, pg. 456 [1])

- Extract-Max and Delete are O(n) worst case
- Constants can be large on some of the operations
- Complicated to implement

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Heaps

Procedure	Binary heap (worst-case)	Binomial heap (worst-case)	Fibonacci heap (amortized)
BUILD-HEAP	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
INSERT	$\Theta(\log n)$	$O(\log n)$	$\Theta(1)$
MAXIMUM	$\Theta(1)$	$O(\log n)$	$\Theta(1)$
EXTRAC-MAX	$\Theta(\log n)$	$\Theta(\log n)$	$O(\log n)$
UNION	$\Theta(n)$	$\Theta(\log n)$	$\Theta(1)$
INCREASE-ELEMENT	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(1)$
DELETE	$\Theta(\log n)$	$\Theta(\log n)$	$O(\log n)$

(adapted from Figure 19.1, pg. 456 [1])

Can we do better?

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