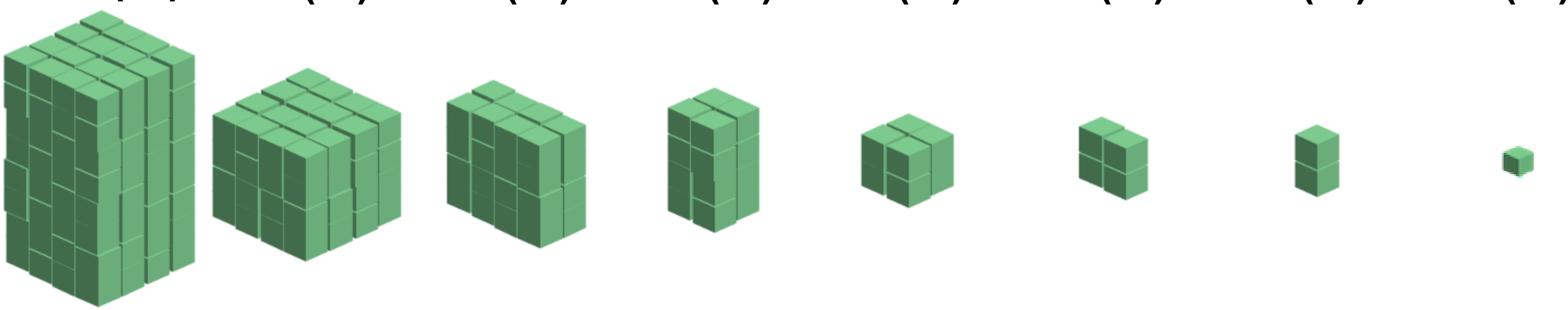
# Lecture 4: Floats

CS 105 Spring 2026

# Review: Representing Integers

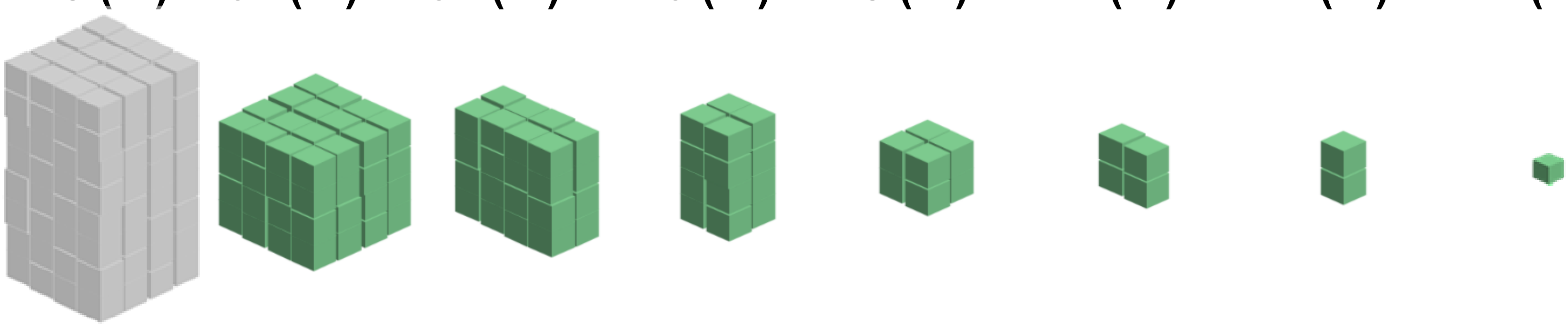
- unsigned:

128 ($2^7$) 64 ($2^6$) 32 ($2^5$) 16 ($2^4$) 8 ($2^3$) 4 ($2^2$) 2 ($2^1$) 1 ($2^0$)

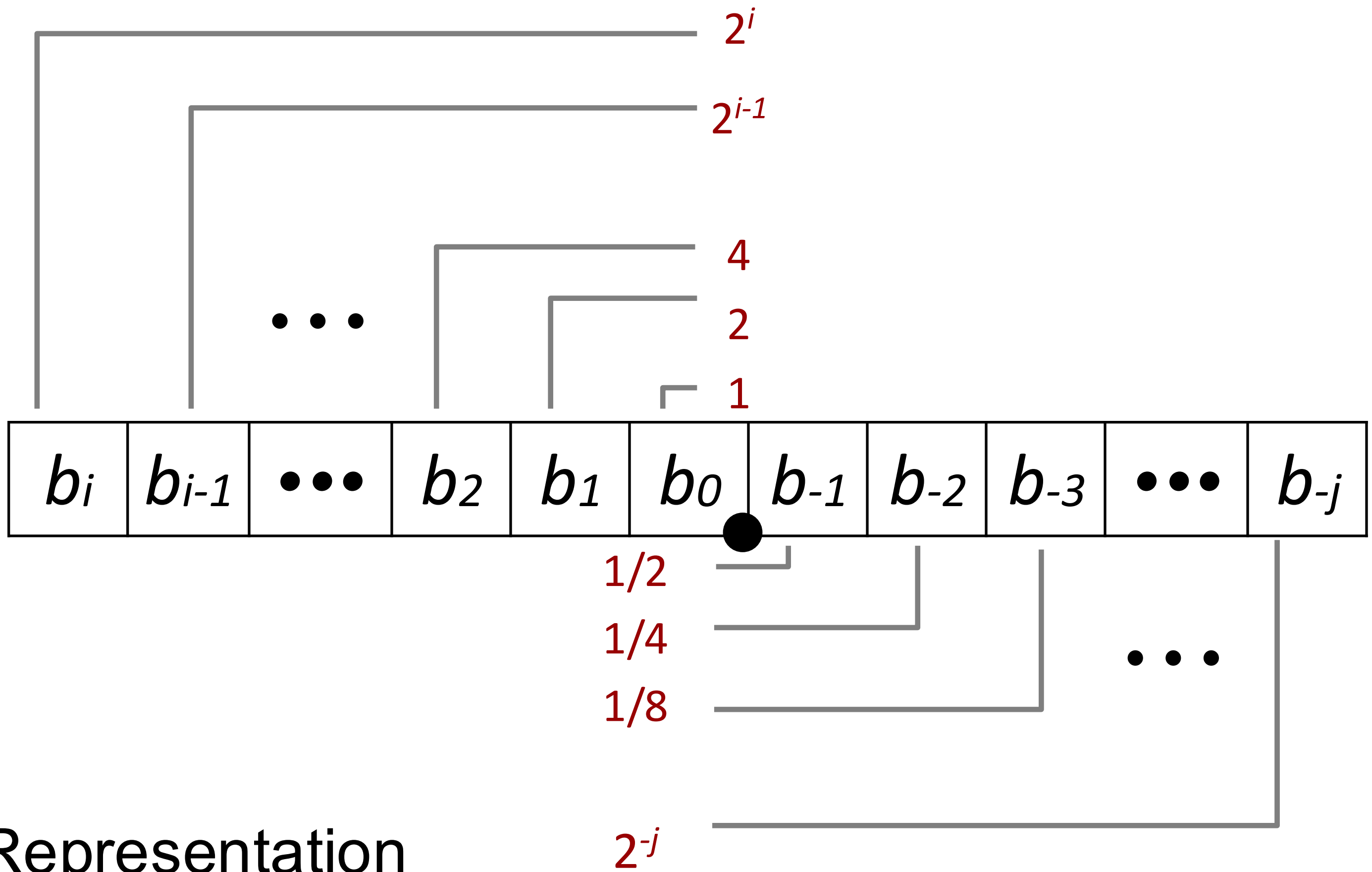


- signed (two's complement aka normal integers):

-128 ($2^7$) 64 ($2^6$) 32 ($2^5$) 16 ($2^4$) 8 ($2^3$) 4 ($2^2$) 2 ($2^1$) 1 ($2^0$)



# Fractional binary numbers




- Representation
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number: $\sum_{k=-j}^{i} (b_k \cdot 2^k)$

# Example: Fractional Binary Numbers

- What is $1001.101_2$?

$$= 8 + 1 + \frac{1}{2} + \frac{1}{8} = 9\,\frac{5}{8} = 9.625$$

- What is the binary representation of 13 9/16?

**1101.1001**

# Exercise 1: Fractional Binary Numbers

- Translate the following fractional numbers to their binary representation
  - 5 3/4
  - 2 7/8
  - 1 7/16

- Translate the following fractional binary numbers to their decimal representation
  - `.011`
  - `.11`
  - `1.1`

# Representable Numbers

- Limitation #1
  - Can only exactly represent numbers of the form $x/2^k$
  - Other rational numbers have repeating bit representations
  - Value — Representation
    - 1/3 — `0.0101010101[01]`…$_2$
    - 1/5 — `0.001100110011[0011]`…$_2$
    - 1/10 — `0.0001100110011[0011]`…$_2$

- Limitation #2
  - Just one setting of binary point within the *w* bits
  - Limited range of numbers (very small values? very large?)

# Floating Point Representation

- Numerical Form: $(-1)^s \cdot M \cdot 2^E$
  - **Sign bit** $s$ determines whether number is negative or positive
  - **Significand** $M$ normally a (binary) fractional value in range [1.0,2.0)
  - **Exponent** $E$ weights value by power of two

- Examples:
  - 1.0
  - -1.25
  - 64
  - .625

# Exercise 2: Floating Point Numbers

- For each of the following numbers, specify a bit s, binary fractional number M in [1.0,2.0) and a binary number E such that the number is equal to $(-1)^s \cdot M \cdot 2^E$
  - 5 3/4
  - 2 7/8
  - -1 1/2
  - -3/4

# Floating Point Representation

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- Encoding:

| $s$ | $\text{exp} = e_{k-1} \ldots e_1 e_0$ | $\text{frac} = f_{n-1} \ldots f_1 f_0$ |
|---|---|---|

  - s is sign bit ***s***
  - exp field encodes $E$ (but is not equal to E)
    - normally $E = e_{k-1} \ldots e_1 e_0 - (2^{k-1} - 1)$ — **bias**
  - frac field encodes ***M*** (but is not equal to M)
    - normally $M = 1.f_{n-1} \ldots f_1 f_0$

Float (32 bits):
- k = 8, n = 23
- bias = 127

Double (64 bits)
- k=11, n = 52
- bias = 1023

# Example: Floats

- What fractional number is represented by the bytes 0x3ec00000? Assume big-endian order.

| $s$ | $\text{exp} = e_{k-1} \ldots e_1 e_0$ | $\text{frac} = f_{n-1} \ldots f_1 f_0$ |
|---|---|---|

- s is sign bit ***s***
- exp field encodes $E$ (but is not equal to E)
  - normally $E = e_{k-1} \ldots e_1 e_0 - (2^{k-1} - 1)$
- frac field encodes ***M*** (but is not equal to M)
  - normally $M = 1.f_{n-1} \ldots f_1 f_0$

Float (32 bits):
- k = 8, n = 23
- bias = 127

$$(-1)^s \cdot M \cdot 2^E$$

**0 | 011 1110 1 | 100 0000 0000 0000 0000 0000**

s=0 exp=125 frac = $10000000000000000000000_2$

s=0 E = -2 M = $1.10000000000000000000000_2 = 1.5_{10}$

$$(-1)^0 \cdot 1.5_{10} \cdot 2^{-2} = 1 \cdot \frac{3}{2} \cdot \frac{1}{4} = \frac{3}{8} = \mathbf{.375_{10}}$$

$$(-1)^0 \cdot 1.1_2 \cdot 2^{-2} = .011_2 = \frac{1}{4} + \frac{1}{8} = \mathbf{.375_{10}}$$

# Exercise 3: Floats

- What fractional number is represented by the bytes 0x423c0000? Assume big-endian order.

| $s$ | $\text{exp} = e_{k-1} \ldots e_1 e_0$ | $\text{frac} = f_{n-1} \ldots f_1 f_0$ |
|---|---|---|

- s is sign bit ***s***
- exp field encodes $E$ (but is not equal to E)
  - normally $E = e_{k-1} \ldots e_1 e_0 - (2^{k-1} - 1)$
- frac field encodes ***M*** (but is not equal to M)
  - normally $M = 1.f_{n-1} \ldots f_1 f_0$

Float (32 bits):
- k = 8, n = 23
- bias = 127

$$(-1)^s \cdot M \cdot 2^E$$


# Limitation so far…

- How do we represent zero?

# Normalized and Denormalized

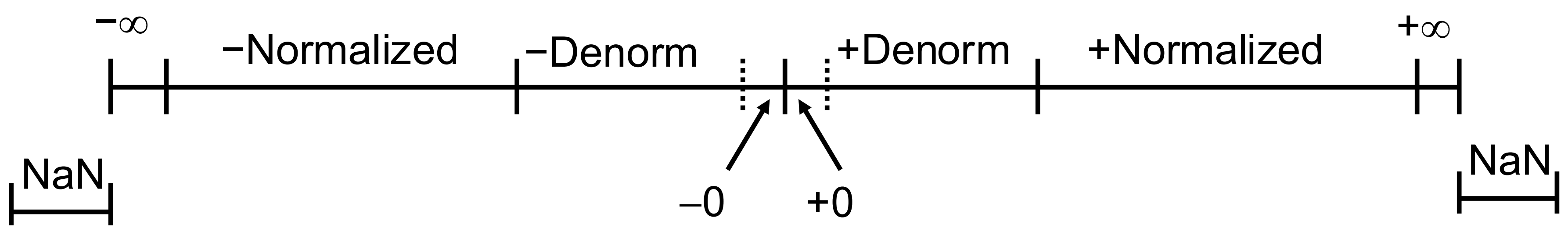

$$(-1)^s \cdot M \cdot 2^E$$

## Normalized Values

- exp is neither all zeros nor all ones (normal case)
- exponent is defined as $\mathrm{E} = e_{k-1} \ldots e_1 e_0 \ - \mathrm{bias}$, where $\mathrm{bias} = 2^{k-1} - 1$ (e.g., 127 for float or 1023 for double)
- significand is defined as $M = 1. f_{n-1} f_{n-2} \ldots f_0$

- ## Denormalized Values
  - exp is either all zeros or all ones
  - if all zeros: $\mathrm{E} = 1 - \mathrm{bias}$ and $M = 0. f_{n-1} f_{n-2} \ldots f_0$
  - if all ones: infinity (if frac is all zeros) or NaN (if frac is non-zero)

# Visualization: Floating Point Encodings



# Limits of Floats

## Denormalized Floats

- Smallest Power of 2
  - M = 0.000…01 = $2^{-23}$
  - val = $2^{-23} \cdot 2^{-126} = 2^{-149}$
- Biggest Power of 2
  - M = 0.100...00 = $2^{-1}$
  - val = $2^{-1} \cdot 2^{-126} = 2^{-127}$

## Normalized Floats

- Smallest Power of 2
  - M = 1.000…00
  - exp = 00000001 = 1
  - val = $1.0 \cdot 2^{1-127} = 2^{-126}$
- Biggest Power of 2
  - M = 1.000...00 = $2^{-1}$
  - exp = 11111110 = 254
  - val = $1.0 \cdot 2^{254-127} = 2^{127}$

| s | exp | frac |
|---|---|---|
| 1 | 8-bits | 23-bits |

# Example: Limits of Floats

- There are big gaps between representable numbers (when exponent is big)

| 0 | 111 1111 0 | 111 1111 1111 1111 1111 1111 |
|---|---|---|
| s=0 | E = 127 | M = $1.11111111111111111111111_2$ |

$$x = 1.11111111111111111111111_2 \cdot 2^{127}$$
$$y = 1.11111111111111111111110_2 \cdot 2^{127}$$

$$x - y = 0.00000000000000000000001_2 \cdot 2^{127} = 1 \cdot 2^{-23} \cdot 2^{127} = \mathbf{2^{104}}$$

# Correctness

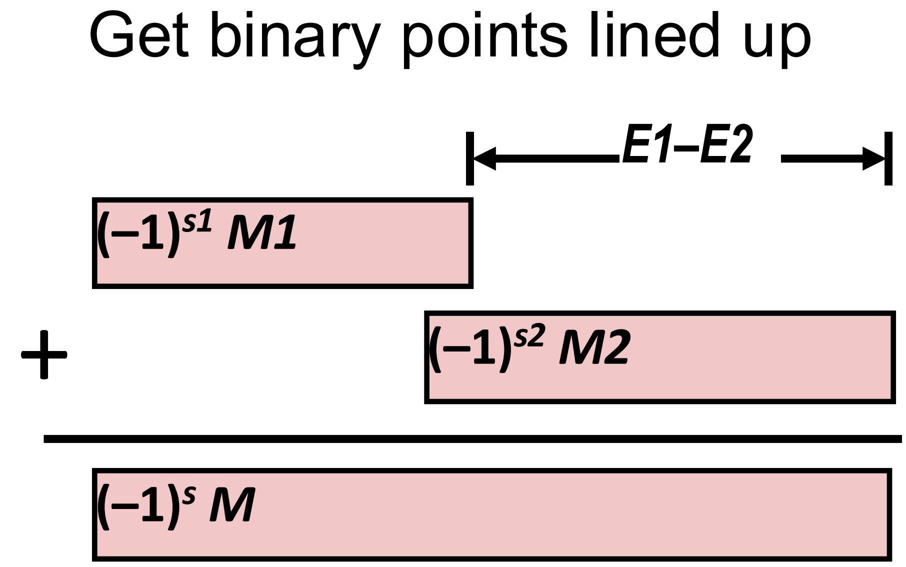
- **Example 1: Is (x + y) + z  =  x + (y + z)?**
  - Ints: Yes!
  - Floats:
    - (2^30 + -2^30) + 3.14 → 3.14
    - 2^30 + (-2^30 + 3.14) → 0.0

# Floating Point Operations

- All of the bitwise and logical operations still work
- Float arithmetic operations done by separate hardware unit (FPU)

# Floating Point Addition

- Float operations done by separate hardware unit (FPU)
- $F_1 + F_2 = (-1)^{s_1} \cdot M_1 \cdot 2^{E_1} + (-1)^{s_1} \cdot M_1 \cdot 2^{E_1}$
  - Assume *E1* >= *E2*




- Exact Result: $(-1)^s \cdot M \cdot 2^E$
  - Sign *s*, significand *M*:
    - Result of signed align & add
  - Exponent *E*: *E1*
- Fixing
  - If *M* ≥ 2, shift *M* right, increment *E*
  - if *M* < 1, shift *M* left *k* positions, decrement *E* by *k*
  - Overflow if *E* out of range
  - Round *M* to fit `frac` precision

# Floating Point Multiplication

- $F_1 \cdot F_2 = (-1)^{s_1} \cdot M_1 \cdot 2^{E_1} \cdot (-1)^{s_1} \cdot M_1 \cdot 2^{E_1}$
- Exact Result: $(-1)^s \cdot M \cdot 2^E$
  - Sign *s*: *s1* ^ *s2*
  - Significand *M*: *M1* x *M2*
  - Exponent *E*: *E1* + *E2*

- Fixing
  - If *M* ≥ 2, shift *M* right, increment *E*
  - If *E* out of range, overflow
  - Round *M* to fit **`frac`** precision

- Implementation
  - Biggest chore is multiplying significands

# Floating Point in C

- C Guarantees Two Levels
  - **`float`** single precision (32 bits)
  - **`double`** double precision (64 bits) ⟸ **This is what most languages call floats!**
- Conversions/Casting
  - Casting between **`int`**, **`float`**, and **`double`** changes bit representation
  - **`double`**/**`float`** → **`int`**
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - **`int`** → **`double`**
    - Exact conversion,
  - **`int`** → **`float`**
    - Will round