Lecture 11: More Sorting and Induction

CS 62

Spring 2019

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Saturday Mentor Hours

- Saturday 10-noon
  - New mentor: Gabe Alzate

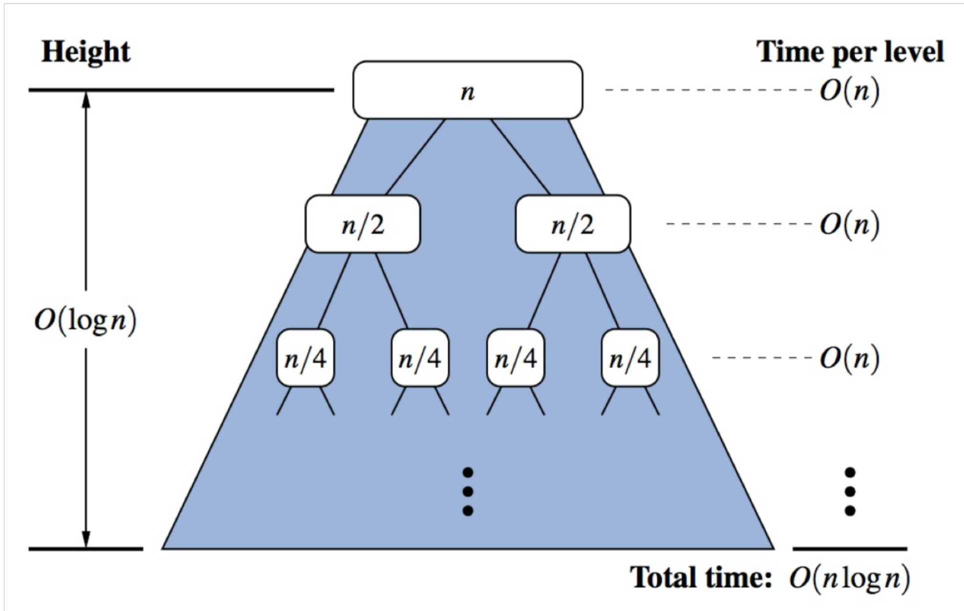
MergeSort

```
/**
* MergeSort   Sorts data >= low and < high
* @param list data to be sorted
* @param low start of the data to be sorted
* @param high end of the data to be sorted (exclusive)
*/
private void mergeSort(int[] data, int low, int high){
   if( high-low > 1 ){
      int mid = low + (high-low)/2;
      mergeSort(data, low, mid);
      mergeSort(data, mid, high);
      merge(data, low, mid, high);
    }
}
```

Complexity

- Claim: $mergeSort$ is $O(n \log n)$
  - where log is base 2
- Merge of two lists of combined size n takes $\leq n - 1$ comparisons.
  - Think of merging [1,3,5,7] and [2,4,6,8]
- If l levels:
  - $n/2^l$=1
  - $n = 2^l$
  - $l = \log n$

- $\log n$ levels
- each taking $O(n)$ operations
- $O(n \ \log n)$ in total



Induction

- Mathematical technique for proving:
  - Mathematical statements over natural numbers
  - Complexity (big-O) of algorithm
  - The correctness of algorithms

- Intimately related to recursion
  - Inductive proofs reference themselves

Induction steps

- Let $P(n)$ be some proposition
  - $P(n)$ should be an assertion

- To prove $P(n)$ is true for all $n \geq 0$
  - (Step 1) Base case: Prove $P(0)$
  - (Step 2) Assume $P(k)$ is true for some $k \geq 0$
  - (Step 3) Use this assumption to prove $P(k+1)$

Practice Examples

- Prove $1 + 2 + \ldots + n = [n(n+1)]/2$ for all $n \geq 1$
- Prove $2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$ for all $n \geq 0$
- Prove $2^n < n!$ for all $n \geq 4$

Recursive Selection Sort

```java
/**
 * @param array array of integers
 * @param startIndex a valid index into array
 */
private static void selectionSortRecursive(int[] array, int startIndex) {
        if(startIndex > 0) {
                //recursively sort array[0,…startIndex-1];
                selectionSortRecursive(array, startIndex-1);
        }
        // find smallest element in array[startIndex…n]
        int smallest = indexOfSmallest(array, startIndex);
        // move smallest element to position startIndex
        swap(array, smallest, startIndex);
}
```

Correctness of Selection Sort

For all $n \geq 0$ after running `selectionSortRecursive(array,n)`, `array[0…n]` has $n + 1$ elements sorted in non-descending order.

$P(n)$: After running `selectionSortRecursive(array,n)`, `array[0...n]` contains the $n + 1$ smallest elements sorted in non-descending order.

Base case: prove $P(0)$

`selectionSortRecursive(array,0)` skips the recursive call, finds the minimum element of `array` and places it at `array[0]`. So the one-element array `array[0…0]` contains the 1st smallest element, which is trivially in non-descending order.

Selection Sort – Induction

- Suppose $P(k)$ is true. i.e. if we call `selectionSortRecursive(array,k)`, then `array[0..k]` will contain the k+1 smallest elements in (non-descending) order

- Prove $P(k+1)$:
  - Call of `selectionSortRecursive(array,k+1)` recursively calls `selectionSortRecursive(array,k)`
  - By induction hypothesis, recursive call of `selectionSortRecursive(array,k)` leaves `array[0...k]` in non-descending order by containing the $k+1$ smallest elements sorted. Selection sort then finds the minimum element in `array[k+1…n]`, which would have to be the $(k+2)\text{nd}$ smallest element overall, and swaps it with the element at `array[k+1]`. Therefore `array[0...k+1]` contains the $k+2$ smallest elements of array in order.

Strong induction

- Sometimes need to assume more than just the previous case, so instead
  - Prove $P(0)$
  - Assumption holds for $P(j)$ for every j $= 0, .., k$ in order to prove $P(k+1)$.

MergeSort

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}
```

Correctness

- $P(n)$: If $high - low = n$ then $mergeSort(data, low, high)$ will result in $data[low\,..high]$ being correctly sorted
- For simplicity, assume $merge$ is correct
- Assume $P(k)\ for\ all\ k < n$, show $P(n)$
- If $n = 0$ or 1 then (correctly) do nothing
- Assume $n > 1$
  - Call $mergeSort(data, low, mid)$ and $mergeSort(data, mid + 1, high)$ where $mid = low + (high - low)/2$.
  - Hence $mid - low < n, high - (mid + 1) < n$
  - By induction $data[low..mid]$ and $data[mid + 1\,..high]$ now sorted.
  - call $merge(data, low, mid, high)$ and, by assumption on $merge$, $data[low\,..high]$ now sorted! Thus $P(n)$ true.

Complexity

- $P(m)$: if data has 2^m elements then $mergesort$ makes $< m \, * \, 2^m$ total comparisons.
- Assume $P(k)\ for\ all\ k\ < m$. Prove $P(m)$
- $P(0)$ is clear. Show $P(m)$
- Sort first half, second half, and then merge
- Each half has size ${2^m}/{2} = 2^{m-1} < 2^m$, so by induction, each takes $< (m-1) * 2^{m-1}$ comparisons
- Therefore total number of comparisons in $mergesort$

$$< (m-1) * 2^{m-1} + (m-1) * 2^{m-1} + (2^m - 1)$$
$$= (m-1) * 2^m + (2^m - 1) = m * 2^m - 1 < m * 2^m$$

- Thus $P(m)$ is true
- If $n = 2^m$ then $mergeSort$ takes $n \log n$ comparisons ($m\ =\ \log n$).