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Structural characterizations of the navigational expressiveness of relation algebras on a tree [☆]

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ABSTRACT

We study the expressiveness on a given document of various fragments of XPath, the core navigational language on XML documents. Viewing these languages as fragments of Tarski's relation algebra, we give characterizations for when a binary relation on the document's nodes (i.e., a set of paths) is definable by an expression in these algebras. In contrast with this "global view" on language semantics, there is also a "local view" where one is interested in the nodes to which one can navigate starting from a particular node. In this view, we characterize when a set of nodes can be defined as the result of applying an expression to a given node. All of these global and local definability results are obtained using a two-step methodology, which consists of first characterizing when two nodes cannot be distinguished by an expression in the language, and then bootstrapping these characterizations to the desired results.

1. Introduction

In this paper, we investigate the expressive power of several basic fragments of Tarski's relation algebra [2] on finite tree-structured graphs. Tarski's algebra is a fundamental tool in the field of algebraic logic which finds various applications in computer science [3–6]. Our investigation is specifically motivated by the role the relation algebra plays in the study of database query languages [7–15], and, more in particular, these query languages restricted to tree-structured graphs [16]. The algebras we consider in this paper correspond to natural fragments of XPath. XPath is a simple language for navigation in XML documents (i.e., a standard syntax for representing node-labeled trees), which is at the heart of standard XML transformation languages and other XML technologies [17]. Keeping in the spirit of XML, we will continue to speak in what follows of trees as "documents" and the algebras we study as "XPath" algebras.

XPath can be viewed as a query language in which an expression associates to every document a binary relation on its nodes representing all navigation paths in the document defined by that expression [9,18,19]. From this query-level

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perspective, several natural semantic issues have been investigated in recent years for various fragments of XPath. These include expressibility, closure properties, and complexity of evaluation [9,10,19–21], as well as decision problems such as satisfiability, containment, and equivalence [22–24].

Alternatively, we can view XPath as a navigational tool on a particular given document, and study expressiveness issues from this document-level perspective. (A similar duality exists in the relational database model, where Bancilhon [25] and Paredaens [26] considered and characterized expressiveness at the instance level, which, subsequently, Chandra and Harel [27] contrasted with expressiveness at the query level.)

In this setting, our goal is to characterize, for various natural fragments of XPath, when a binary relation on the nodes of a given document (i.e., a set of navigation paths) is definable by an expression in the fragment.

To achieve this goal, we develop a robust two-step methodology. The first step consists of characterizing when two nodes in a document cannot be distinguished by an expression in the fragment under consideration. It turns out for those fragments we consider that this notion of expression equivalence of nodes is equivalent to an appropriate generalization of the classic notion of bisimilarity [28]. The second step of our methodology then consists of bootstrapping this result to a characterization for when a binary relation on the nodes of a given document is definable by an expression in the fragment.

We refer to this perspective on the semantics of XPath at the document level as the "global view." In contrast with this global view, there is also a "local view" which we consider. In this view, one is only interested in the nodes to which one can navigate starting from a particular given node in the document under consideration. From this perspective, a set of nodes of that document can be seen as the end points of a set of paths starting at the given node. For each of the XPath fragments considered, we characterize when such a set represents the set of *all* paths starting at the given node defined by some expression in the fragment. These characterizations are derived from the corresponding characterizations in the "global view," and turn out to be particularly elegant in the important special case where the starting node is the root.

In this paper, we study several natural XPath fragments. The most expressive among them is the *XPath algebra* which permits the self, parent, and child operators, predicates, compositions, and the boolean operators union, intersection, and difference. (Since we work at the document level, i.e., the document is given, there is no need to include the ancestor and descendant operators as primitives.) We also consider the *core XPath algebra*, which is the XPath algebra without intersection and difference at the expression level. The core XPath algebra is the adaptation to our setting of Core XPath of Gottlob et al. [18,21,29]. Of both of these algebras, we also consider various "downward" and "upward" fragments without the parent and child operator, respectively. We also study "positive" variants of all the fragments considered, without the difference operator.

Our strategy is to introduce and characterize generalizations of each of these practical fragments, towards a broader perspective on relation algebras on trees. These generalizations are based on a simple notion of path counting, a feature which also appears in XPath.

The robustness of the characterizations provided in this paper is further strengthened by their feasibility. As discussed in Section 9, the global and local definability problems for each of the XPath fragments are decidable in polynomial time. This feasibility hints towards efficient partitioning and reduction techniques on both the set of nodes and the set of paths in a document. Such techniques may be applied fruitfully towards, e.g., document compression [30], access control [31], and designing indexes for efficient query processing [11,32,33].

We proceed in the paper as follows. In Section 2, we formally define documents and the algebras, and then in Section 3, we define a notion of "signatures" which will be essential in the sequel. In Section 4, we define the semantic and syntactic notions of node distinguishability necessary to obtain our desired structural characterizations. In the balance of the paper, we apply our two-step methodology to link semantic expression equivalence in the languages to appropriate structural syntactic equivalence notions. In particular, we give structural characterizations, under both the global and local views,

- of "strictly" (Section 5) and "weakly" (Section 6) downward languages, and their positive variants;
- of upward languages and their positive variants (Section 7); and
- of languages with both downward and upward navigation, and their positive variants (Section 8).

Along the way, we also establish the equivalence of some of these fragments, using the structural characterizations obtained. We conclude in Section 9 with a discussion of some ramifications of our results and directions for further study.

2. Documents and navigation

In this paper, we are interested in navigating over documents in the form of unordered labeled trees. Formally, we denote such a document as $D = (V, Ed, r, \lambda)$, with D the document name, V the set of nodes of the tree, Ed the set of edges of the tree, r the root of the tree, and $\lambda : V \to \mathcal{L}$ a function assigning to each node a label from some infinite set of labels $\mathcal{L}$.

Example 2.1. Fig. 1 shows an example of a document that will be used throughout the paper. Here, $r = v_1$ is the root of the tree with label $\lambda(v_1) = a$.

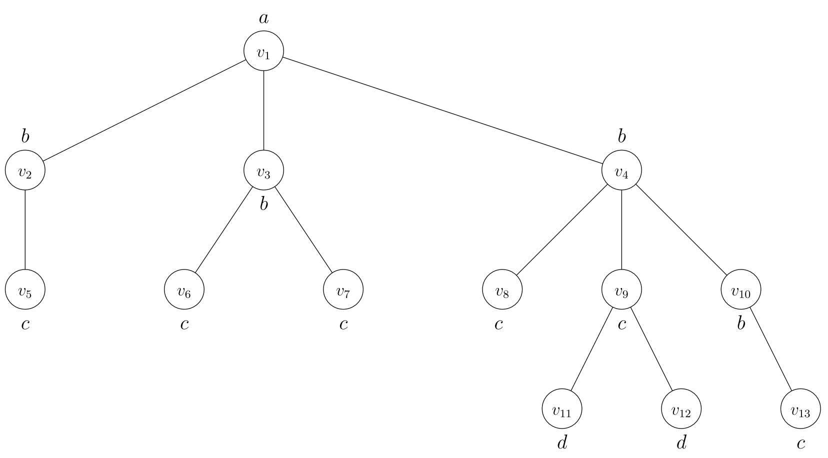


Fig. 1. Example document.

Table 1
Binary operations on documents. The left column shows the syntax of the operation, and the right column its semantics, when the operator is applied to a given document $D = (V, Ed, r, \lambda)$. Below, ℓ is a label in $\mathcal{L}$ and $k \geq 1$ a natural number. Furthermore, in the recursive definitions, e, e_1, and e_2 represent expressions built with the operations.

Operator	*Operator*(D)
$\emptyset$	$\emptyset$
ε	$\{(v, v) \mid v \in V\}$
$\hat{\ell}$	$\{(v, v) \mid v \in V \;\&\; \lambda(v) = \ell\}$
$\downarrow$	Ed
$\uparrow$	Ed^{-1}
$\pi_1(e)$	$\{(v, v) \mid (\exists w)(v, w) \in e(D)\}$
$\pi_2(e)$	$\{(w, w) \mid (\exists v)(v, w) \in e(D)\}$
e^{-1}	$e(D)^{-1}$
$\text{ch}_{\geq k}(e)$	$\{(v, v) \mid v \in V \;\&\; \lvert\{w \mid (v, w) \in Ed \;\&\; (w, w) \in \pi_1(e)(D)\}\rvert \geq k\}$
e_1/e_2	$\{(u, w) \mid (\exists v)((u, v) \in e_1(D) \;\&\; (v, w) \in e_2(D))\}$
$e_1 \cup e_2$	$e_1(D) \cup e_2(D)$
$e_1 \cap e_2$	$e_1(D) \cap e_2(D)$
$e_1 - e_2$	$e_1(D) - e_2(D)$

We next define a set of operations on documents, as tabulated in Table 1. The left column shows the syntax of the operation, and the right column its semantics, given a document $D = (V, Ed, r, \lambda)$. Notice that, in each case, the result is a binary relation on the nodes of the document.

The *basic algebra*, denoted $\mathcal{X}$, is the language consisting of all expressions built from $\emptyset$, ε, $\hat{\ell}$ with $\ell \in \mathcal{L}$, composition ("/"), and union ("$\cup$").[1] The basic algebra $\mathcal{X}$ can be extended by adding some of the other operations in Table 1, which we call *nonbasic*. If E is a set of nonbasic operations, then $\mathcal{X}(E)$ denotes the algebra obtained by adding the operations in E to the basic algebra $\mathcal{X}$.

Notice that we do not consider transitive closure operations such as the descendant ("$\downarrow^*$") or ancestor ("$\uparrow^*$") operations of XPath. The reason for this is that, in this paper, we do not reason at the query level but only consider navigation within a given document.[2]

Example 2.2. Consider the document D in Fig. 1. Let e be the expression $\uparrow/\pi_1(\downarrow/\hat{b}/\downarrow/\hat{c}) - \text{ch}_{\geq 2}(\varepsilon)/\uparrow$ in the language $\mathcal{X}(\downarrow, \uparrow, \pi_1, \text{ch}_{\geq 2}(.), -)$ (or, for that matter, in any language $\mathcal{X}(E)$ with $\{\downarrow, \uparrow, \pi_1, \text{ch}_{\geq 2}(.), -\} \subseteq E$). Then, $e(D) = \{(v_2, v_1), (v_8, v_4), (v_{10}, v_4)\}$.

Not all the above operations are primitive, however. For instance, intersection ("$\cap$") is expressible as soon a set difference ("$-$") is expressible, since, for any two sets A and B, $A \cap B = A - (A - B)$. Even more eliminations are possible in the following setting.

[1] When writing expressions, we assume that unary operations take precedence over binary operations, and that composition takes precedence over the set operations.

[2] Indeed, if d is the height of the given document, then, on that document, e.g., $\downarrow^*$ is equivalent to $\bigcup_{i=1}^{d} \downarrow^i$, where $\downarrow^i$ denotes i-fold composition of $\downarrow$.

Proposition 2.3. *Let E be a set of nonbasic operations containing set difference ("$-$") or intersection ("$\cap$") for which "$\downarrow$" and "$\uparrow$" are both contained in E or both not contained in E. Then, for each expression e in $\mathcal{X}(E)$, there is an equivalent expression in $\mathcal{X}(E - \{\pi_1, \pi_2, {}^{-1}\})$.*

Proof. First, we eliminate both projections using the identities

$$\pi_1(e) = (e/e^{-1}) \cap \varepsilon;$$
$$\pi_2(e) = (e^{-1}/e) \cap \varepsilon.$$

To eliminate inverse ("$.^{-1}$"), we first observe that we can propagate this operation down to the level of the atomic operations $\emptyset$, ε, $\hat{\ell}$ ($\ell \in \mathcal{L}$), $\uparrow$, and $\downarrow$. Of these, inverse has only an effect on $\uparrow$ and $\downarrow$. It suffices now to note that $\uparrow^{-1}$ and $\downarrow^{-1}$ are equivalent to $\downarrow$ and $\uparrow$, respectively. $\square$

Notice that in a language with both upward ("$\uparrow$") and downward ("$\downarrow$") navigation, the identities $\pi_1(e)(D) = \pi_2(e^{-1})(D)$ and $\pi_2(e)(D) = \pi_1(e^{-1})(D)$ imply that one projection operation can be eliminated in favor of the other. Hence, it does not make sense to consider them separately in this context.

Some counting operations ("$\mathrm{ch}_{\geq k}(e)$") can also be simulated. One can easily verify the following.

Proposition 2.4. *Let $D = (V, Ed, r, \lambda)$ be a document. Then,*

1. $\mathrm{ch}_{\geq 1}(e)(D) = \pi_1(\downarrow/e)(D)$*;*
2. $\mathrm{ch}_{\geq 2}(e)(D) = \pi_1(\downarrow/(\pi_1(e)/\uparrow/\downarrow/\pi_1(e) - \varepsilon))(D)$*; and*
3. $\mathrm{ch}_{\geq 3}(e)(D) = \pi_1(\downarrow/((\pi_1(e)/\uparrow/\downarrow/\pi_1(e) - \varepsilon)/(\pi_1(e)/\uparrow/\downarrow/\pi_1(e) - \varepsilon) - \varepsilon))(D)$.

Example 2.5. Consider again the expression $e := \uparrow/\pi_1(\downarrow/\hat{b}/\downarrow/\hat{c}) - \mathrm{ch}_{\geq 2}(\varepsilon)/\uparrow$ of Example 2.2. Using Proposition 2.4, and making some straightforward simplifications, we can rewrite e as $\uparrow/\pi_1(\downarrow/\hat{b}/\downarrow/\hat{c}) - \pi_1(\downarrow/(\uparrow/\downarrow - \varepsilon))/\uparrow$, an expression of $\mathcal{X}(\downarrow, \uparrow, \pi_1, -)$. Using Proposition 2.3 and the techniques exhibited in its proof, we can further rewrite this as $\uparrow/(\downarrow/\hat{b}/\downarrow/\hat{c}/\uparrow/\downarrow) - \downarrow/((\uparrow/\downarrow - \varepsilon)/(\uparrow/\downarrow - \varepsilon) \cap \varepsilon)/\uparrow$, an expression in $\mathcal{X}(\downarrow, \uparrow, \cap, -)$. Finally, we invite the reader to verify that e can also be rewritten as $\pi_1(\varepsilon - \pi_1(\downarrow/(\uparrow/\downarrow - \varepsilon)))/\uparrow/\pi_1(\downarrow/\hat{b}/\downarrow/\hat{c})$ also an expression of $\mathcal{X}(\downarrow, \uparrow, \pi_1, -)$.

We shall call the language $\mathcal{X}(\downarrow, \uparrow, \pi_1, \pi_2, .^{-1}, \cap, -)$, which by Proposition 2.3 is equivalent to $\mathcal{X}(\downarrow, \uparrow, -)$, the *XPath algebra*, because, on a given document, it is equivalent to basic XPath [17].[3]

Besides the *standard languages* $\mathcal{X}(E)$, with E a set of nonbasic operations, we also consider the so-called *core languages* $\mathcal{C}(E)$. More concretely, $\mathcal{C}(E)$ is defined recursively in the same way as $\mathcal{X}(E - \{\cap, -\})$, except that in expressions of the form $\pi_1(f)$, and $\pi_2(f)$, f may be a boolean combination of expressions of the language using union and the operations in $E \cap \{\cap, -\}$, rather than just an expression of the language.

The above terminology is inspired by the fact that $\mathcal{C}(\downarrow, \uparrow, \pi_1, \pi_2, -, \cap)$, the language which we call the *core XPath algebra*, is the adaptation to our setting of Core XPath of Gottlob and Koch [18].

Example 2.6. Continuing with Example 2.5, we consider again the expression $e := \uparrow/\pi_1(\downarrow/\hat{b}/\downarrow/\hat{c}) - \mathrm{ch}_{\geq 2}(\varepsilon)/\uparrow$ of Example 2.2. Obviously, there is no core language of which e is an expression, as set difference ("$-$") occurs at the outer level, and not in a subexpression f which in turn is embedded in a subexpression of the form $\pi_1(f)$ or $\pi_2(f)$. However, in Example 2.5, the expression e has been shown to be equivalent to $\pi_1(\varepsilon - \pi_1(\downarrow/(\uparrow/\downarrow - \varepsilon)))/\uparrow/\pi_1(\downarrow/\hat{b}/\downarrow/\hat{c})$ which is an expression of $\mathcal{C}(\downarrow, \uparrow, \pi_1, -, \cap)$, and hence also of the core XPath algebra.

Given a set of nonbasic operators E, an expression in $\mathcal{X}(E)$ can in general *not* be converted to an equivalent expression in $\mathcal{C}(E)$, however, as will follow from the results of this paper, although there are exceptions (see Theorem 5.19).

Table 2 gives an overview of the relation algebra fragments we consider.

To conclude this section, we observe that, given a document and an expression, we have defined the semantics of that expression as a binary relation over the nodes of the document, i.e., as a set of pair of nodes. From the perspective of navigation, however, it is useful to be able to say that an expression allows one to navigate from one node of the document to another. For this purpose, we introduce the following notation.

Definition 2.7. Let e be an arbitrary expression, and let $D = (V, Ed, r, \lambda)$ be a document. For $v \in V$, $e(D)(v) := \{w \mid (v, w) \in e(D)\}$.

[3] Note that the XPath algebra corresponds to the (full) relation algebra of Tarski [2], adapted to our setting (cf. [10]).

Table 2
Languages studied in this paper.

Language	Relation Algebra Fragment
Strictly downward (core) XPath algebra with counting up to k	$\mathcal{X}(\downarrow,\pi_1,\mathrm{ch}_{\geq 1}(.),\ldots,\mathrm{ch}_{\geq k}(.),-)=\mathcal{C}(\downarrow,\pi_1,\mathrm{ch}_{\geq 1}(.),\ldots,\mathrm{ch}_{\geq k}(.),-)$
Strictly downward (core) XPath algebra	$\mathcal{X}(\downarrow,\pi_1,-)=\mathcal{C}(\downarrow,\pi_1,-)$
Strictly downward positive (core) XPath algebra	$\mathcal{X}(\downarrow,\pi_1,\cap)=\mathcal{X}(\downarrow,\pi_1)=\mathcal{C}(\downarrow,\pi_1,\cap)=\mathcal{C}(\downarrow,\pi_1)$
Weakly downward (core) XPath algebra with counting up to k	$\mathcal{X}(\downarrow,\pi_1,\pi_2,\mathrm{ch}_{\geq 1}(.),\ldots,\mathrm{ch}_{\geq k}(.),-)=\mathcal{C}(\downarrow,\pi_1,\pi_2,\mathrm{ch}_{\geq 1}(.),\ldots,\mathrm{ch}_{\geq k}(.),-)$
Weakly downward (core) XPath algebra	$\mathcal{X}(\downarrow,\pi_1,\pi_2,-)=\mathcal{C}(\downarrow,\pi_1,\pi_2,-)$
Weakly downward positive (core) XPath algebra	$\mathcal{X}(\downarrow,\pi_1,\pi_2,\cap)=\mathcal{X}(\downarrow,\pi_1,\pi_2)=\mathcal{C}(\downarrow,\pi_1,\pi_2,\cap)=\mathcal{C}(\downarrow,\pi_1,\pi_2)$
Strictly upward (core) XPath algebra	$\mathcal{X}(\uparrow,\pi_1,-)=\mathcal{C}(\uparrow,\pi_1,-)$
Strictly upward positive (core) XPath algebra	$\mathcal{X}(\uparrow,\pi_1,\cap)=\mathcal{X}(\uparrow,\pi_1)=\mathcal{C}(\uparrow,\pi_1,\cap)=\mathcal{C}(\uparrow,\pi_1)$
Weakly upward languages	See Section 7.2
XPath algebra with counting up to k	$\mathcal{X}(\downarrow,\uparrow,\mathrm{ch}_{\geq 1}(.),\ldots,\mathrm{ch}_{\geq k}(.),-)$
XPath algebra	$\mathcal{X}(\downarrow,\uparrow,\pi_1,\pi_2,.^{-1},\cap,-)=\mathcal{X}(\downarrow,\uparrow,-)$
Core XPath algebra with counting up to k	$\mathcal{C}(\downarrow,\uparrow,\pi_1,\pi_2,\mathrm{ch}_{\geq 1}(.),\ldots,\mathrm{ch}_{\geq k}(.),-)$
Core XPath algebra	$\mathcal{C}(\downarrow,\uparrow,\pi_1,\pi_2,-,\cap)$
Positive (core) XPath algebra ([34])	$\mathcal{X}(\downarrow,\uparrow,\cap)=\mathcal{X}(\downarrow,\uparrow,\pi_1,\pi_2)=\mathcal{C}(\downarrow,\uparrow,\pi_1,\pi_2,\cap)$

Definition 2.7 reflects the "local" perspective of an expression working on particular nodes of a document, rather than the "global" perspective of working on an entire document.

Example 2.8. Consider again the expression $e := \uparrow/\pi_1(\downarrow/\hat{b}/\downarrow/\hat{c}) - \mathrm{ch}_{\geq 2}(\varepsilon)/\uparrow$ of Example 2.2. We have established that, for the document D in Fig. 1, $e(D)=\{(v_2,v_1),(v_8,v_4),(v_{10},v_4)\}$. Hence, $e(D)(v_8)=\{v_4\}$ and $e(D)(v_1)=\emptyset$.

3. Signatures

Given a pair of nodes in a document, there is a unique path in that document (not taking into account the direction of the edges) to navigate from the first to the second node, in general by going a few steps upward in the tree, and then going a few steps downward. We call this the *signature* of that pair of nodes, and shall formally represent it by an expression in $\mathcal{X}(\downarrow,\uparrow)$.

Definition 3.1. Let $D=(V,Ed,r,\lambda)$ be a document, and let $v,w\in V$. The *signature* of the pair (v,w), denoted $\mathrm{sig}(v,w)$, is the expression in $\mathcal{X}(\downarrow,\uparrow)$ that is recursively defined as follows:

- if $v=w$, then $\mathrm{sig}(v,w):=\varepsilon$;
- if v is an ancestor of w, and z is the child of v on the path from v to w, then $\mathrm{sig}(v,w):=\downarrow/\mathrm{sig}(z,w)$;
- otherwise,[4] if z is the parent of v, then $\mathrm{sig}(v,w):=\uparrow/\mathrm{sig}(z,w)$.

Given nodes v and w of a document $D=(V,Ed,r,\lambda)$, we denote by $\mathrm{top}(v,w)$ the unique node on the undirected path from v to w that is an ancestor of both v and w. Clearly, $\mathrm{sig}(v,w)=\mathrm{sig}(v,\mathrm{top}(v,w))/\mathrm{sig}(\mathrm{top}(v,w),w)=\uparrow^m/\downarrow^n$, where m, respectively n, is the distance from $\mathrm{top}(v,w)$ to v, respectively w; and, for an expression e and a natural number $i\geq 1$, e^i denotes the i-fold composition of e.[5] (We put $e^0:=\varepsilon$.)

The signature of a pair of nodes of a document can be seen as a description of the unique path connecting these nodes, but also as an expression that can be applied to the given document. We shall often exploit this duality.

Example 3.2. For the document D in Fig. 1, $\mathrm{sig}(v_1,v_1)=\varepsilon$, $\mathrm{sig}(v_1,v_2)=\downarrow$, $\mathrm{sig}(v_6,v_4)=\uparrow^2/\downarrow$, and $\mathrm{sig}(v_{11},v_5)=\uparrow^3/\downarrow^2$. We have that

$$\begin{aligned}\mathrm{sig}(v_{11},v_5)(D)=\{&(v_{11},v_5),(v_{12},v_5),(v_{13},v_5),(v_{11},v_6),(v_{12},v_6),(v_{13},v_6),\\&(v_{11},v_7),(v_{12},v_7),(v_{13},v_7),(v_{11},v_8),(v_{12},v_8),(v_{13},v_8),\\&(v_{11},v_9),(v_{12},v_9),(v_{13},v_9),(v_{11},v_{10}),(v_{12},v_{10}),(v_{13},v_{10})\}.\end{aligned}$$

Notice that not each pair in the result has the same signature as (v_{11},v_5). For instance, $\mathrm{sig}(v_{11},v_8)=\uparrow^2/\downarrow$ and $\mathrm{sig}(v_{11},v_9)=\uparrow$.

Now, let (v_1,w_1) and (v_2,w_2) be two pairs of nodes in a document $D=(V,Ed,r,\lambda)$. We say that (v_1,w_1) *subsumes* (v_2,w_2), denoted $(v_1,w_1)\gtrsim(v_2,w_2)$, if $(v_2,w_2)\in\mathrm{sig}(v_1,w_1)(D)$. We say that (v_1,w_1) are (v_2,w_2) *congruent*, denoted

[4] In particular, $v\neq r$.

[5] Here, and elsewhere in this paper, equality between expressions must be interpreted at the semantic and not at the syntactic level, i.e., for two expressions e_1 and e_2 in one of the languages considered here, $e_1=e_2$ means that, for each document D, $e_1(D)=e_2(D)$.

$(v_1, w_1) \cong (v_2, w_2)$, if $(v_1, w_1) \gtrsim (v_2, w_2)$ and $(v_2, w_2) \gtrsim (v_1, w_1)$. It can be easily seen that, in this case, $\text{sig}(v_1, w_1) = \text{sig}(v_2, w_2)$. Informally speaking, the path from v_1 to w_1 has then the same shape as the path from v_2 to w_2.

Example 3.3. Consider again Example 3.2. Clearly, (v_{11}, v_5) subsumes each pair of nodes in $\text{sig}(v_{11}, v_5)(D)$, e.g., $(v_{11}, v_5) \gtrsim (v_{12}, v_6)$ and $(v_{11}, v_5) \gtrsim (v_{12}, v_9)$. Notice that also $(v_{12}, v_6) \gtrsim (v_{11}, v_5)$, and hence $(v_{11}, v_5) \cong (v_{12}, v_6)$. However, $(v_{12}, v_9) \not\gtrsim (v_{11}, v_5)$. Hence, these pairs are not congruent.

By definition, subsumption is captured by the "sig" expression. One may wonder if there also exists an expression that precisely captures congruence. This is the case in the following situations.

Proposition 3.4. *Let $D = (V, Ed, r, \lambda)$ be a document and let $v_1, v_2, w_1, w_2 \in V$. Then,*

1. *if v_1 is an ancestor of w_1 or vice versa, $(v_1, w_1) \cong (v_2, w_2)$ if and only if $(v_2, w_2) \in \text{sig}(v_1, w_1)(D)$;*
2. *otherwise, let $\text{sig}(v_1, w_1) = \uparrow^m/\downarrow^n$. Then, as $m \geq 1$ and $n \geq 1$, $(v_1, w_1) \cong (v_2, w_2)$ if and only if $(v_2, w_2) \in \uparrow^m/\downarrow^n - \uparrow^{m-1}/\downarrow^{n-1}(D)$.*

Proof. We only prove the "if" part of Property 2. Let $t_2 := \uparrow^m(D)(v_2)$. Since $w_2 \in \downarrow^n(D)(t_2)$, t_2 is a common ancestor of v_2 and w_2. Let v_2' and w_2' be the children of t_2 on the path to v_2 and w_2, respectively. If $v_2' = w_2'$, then $(v_2, w_2) \in \uparrow^{m-1}/\downarrow^{n-1}(D)$, a contradiction. Hence, $v_2' \neq w_2'$ and $t_2 = \text{top}(v_2, w_2)$, and $\text{sig}(v_2, w_2) = \uparrow^m/\downarrow^n = \text{sig}(v_1, w_1)$. □

For later use, but also because of their independent interest, we finally note the following fundamental properties of subsumption and congruence.

Proposition 3.5. *Let $D = (V, Ed, r, \lambda)$ be a document, and let $v, w, v_1, w_1, z_1, v_2, w_2, z_2 \in V$. Then the following properties hold.*

1. $(v, v) \gtrsim (w, w)$.
2. $(v_1, w_1) \gtrsim (v_2, w_2)$ *implies that* $(w_1, v_1) \gtrsim (w_2, v_2)$.
3. *If $\text{top}(v_1, z_1)$ is also an ancestor of w_1, then $(v_1, w_1) \gtrsim (v_2, w_2)$ and $(w_1, z_1) \gtrsim (w_2, z_2)$ imply that $(v_1, z_1) \gtrsim (v_2, z_2)$.*
4. *All properties above also hold when subsumption is replaced by congruence, provided that, in item 3, the condition "$\text{top}(v_2, z_2)$ is also an ancestor of w_2" is added.*

Proof. All properties are straightforward, except for Property 3. So, assume that $(v_1, w_1) \gtrsim (v_2, w_2)$ and $(v_1, z_1) \gtrsim (v_2, z_2)$. Hence, $(v_2, w_2) \in \text{sig}(v_1, w_1)(D)$ and $(v_2, z_2) \in \text{sig}(v_1, z_1)(D)$, as a consequence of which

$$(v_2, z_2) \in \text{sig}(v_1, w_1)/\,\text{sig}(w_1, z_1)(D).$$

For the sake of abbreviation, let $t_1 := \text{top}(v_1, w_1)$ and $u_1 := \text{top}(w_1, z_1)$. Using these nodes, we can write

$$\text{sig}(v_1, w_1)/\,\text{sig}(w_1, z_1) = \text{sig}(v_1, t_1)/\,\text{sig}(t_1, w_1)/\,\text{sig}(w_1, u_1)/\,\text{sig}(u_1, z_1),$$

which is equal to $\text{sig}(v_1, s_1)/\,\text{sig}(s_1, z_1)$, where s_1 is the higher of t_1 and u_1 in D. Notice that s_1 is a common ancestor of v_1 and z_1, as a consequence of which it is also an ancestor of $\text{top}(v_1, z_1)$, the least common ancestor of v_1 and z_1. By assumption, $\text{top}(v_1, z_1)$ is a common ancestor of v_1, w_1, and z_1, and hence also of $\text{top}(v_1, w_1)$ and $\text{top}(w_1, z_1)$, the highest of which is s_1. Thus, $s_1 = \text{top}(v_1, z_1)$, and, therefore, $\text{sig}(v_1, s_1)/\,\text{sig}(s_1, z_1) = \text{sig}(v_1, z_1)$. In summary, $(v_2, z_2) \in \text{sig}(v_1, z_1)(D)$, and hence $(v_1, z_1) \gtrsim (v_2, z_2)$. □

Observe that the condition in Proposition 3.5, (3), is necessary for that part of the proposition to hold, as shown by the following counterexample.

Example 3.6. Consider the document in Fig. 2. Labels have been omitted, because they are not relevant in this discussion. (We assume all nodes have the same label.) Observe that $(v_1, w_1) \cong (v_2, w_2)$ and $(w_1, z_1) \cong (w_2, z_2)$. However, $\text{top}(v_1, z_1)$ is *not* an ancestor of w_1, hence, Proposition 3.5, (3), is *not* applicable. We see that, indeed, (v_1, z_1) does *not* subsume (v_2, z_2), let alone that (v_1, z_1) and (v_2, z_2) would be congruent.

4. Distinguishability of nodes in a document

We wish to link the distinguishing power of a navigational language on a document to syntactic conditions which can readily be verified on that document. As argued before, the action of an expression on a document can be interpreted as (1) returning pairs of nodes, or (2) given a node, returning the set of nodes that can be reached from that node. We shall refer to the first interpretation as the *pairs semantics*, and to the second interpretation as the *node semantics*. In this section, we propose suitable semantic and syntactic notions of distinguishability for the node semantics.

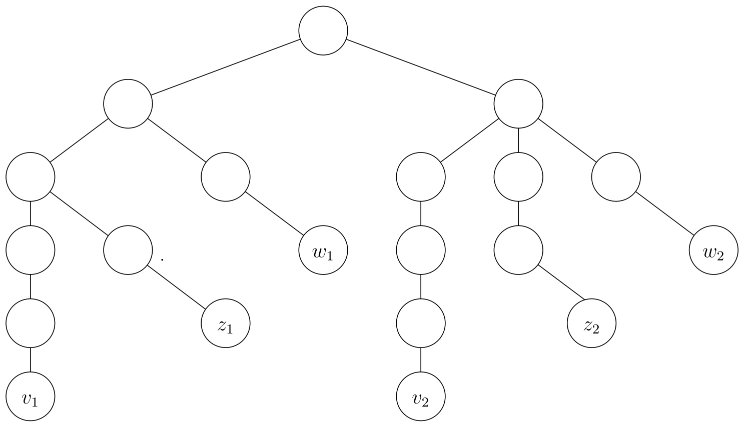


Fig. 2. Document of Example 3.6.

4.1. Distinguishability of nodes at the semantic level

We propose the following distinguishability criterion based on the emptiness or nonemptiness of the set of nodes that can be reached by applying an arbitrary expression of the language under consideration.

Definition 4.1. Let L be one of the languages considered in Section 2. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. Then,

1. v_1 and v_2 are *expression-related*, denoted $v_1 \geq_{\text{exp}} v_2$, if, for each expression e in L, $e(D)(v_1) \neq \emptyset$ implies $e(D)(v_2) \neq \emptyset$; and
2. v_1 and v_2 are *expression-equivalent*, denoted $v_1 \equiv_{\text{exp}} v_2$, if $v_1 \geq_{\text{exp}} v_2$ and $v_2 \geq_{\text{exp}} v_1$.

In principle, we should have reflected the language under consideration in the notation for expression-equivalence. As the language under consideration will always be clear from the context, we chose not to do so in order to avoid overloaded notation.

The following observation is useful.

Proposition 4.2. *Let E be a set of nonbasic operations containing first projection ("π_1") and set difference ("$-$"). Consider expression-equivalence with respect to either $\mathcal{X}(E)$ or $\mathcal{C}(E)$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. Then, $v_1 \equiv_{\text{exp}} v_2$ if and only if $v_1 \geq_{\text{exp}} v_2$.*

Proof. Assume that $v_2 \not\geq_{\text{exp}} v_1$. Then there exists an expression f in $\mathcal{X}(E)$ or $\mathcal{C}(E)$ such that $f(D)(v_2) \neq \emptyset$ and $f(D)(v_1) = \emptyset$. Now consider $e := \pi_1(\varepsilon - \pi_1(f))$, which is also an expression of $\mathcal{X}(E)$, respectively $\mathcal{C}(E)$. Clearly, $e(D)(v_2) = \emptyset$ and $e(D)(v_1) \neq \emptyset$, hence $v_1 \not\geq_{\text{exp}} v_2$. By contraposition, $v_1 \geq_{\text{exp}} v_2$ implies $v_2 \geq_{\text{exp}} v_1$, and hence also $v_1 \equiv_{\text{exp}} v_2$. □

4.2. Distinguishability of nodes at the syntactic level

Our syntactic criterion of distinguishability is based on the similarity of the documents locally around the nodes under consideration. In order to decide this similarity, we shall consider a hierarchy for the degree of coarseness by which we compare the environments of those nodes. We shall also consider variants for the cases where from the given nodes of the document we (1) only look downward; (2) only look upward; or (3) look in both directions.

4.2.1. Downward distinguishability

For the downward case, we consider the following syntactic notions of node distinguishability. They are defined recursively on the height of the first node.

Definition 4.3. Let $D = (V, Ed, r, \lambda)$ be a document, let $v_1, v_2 \in V$, and let $k \geq 1$. Then, v_1 and v_2 are *downward-k-equivalent*, denoted $v_1 \equiv_{\downarrow}^{k} v_2$, if

1. $\lambda(v_1) = \lambda(v_2)$;
2. for each child w_1 of v_1, there exists a child w_2 of v_2 such that $w_1 \equiv_{\downarrow}^{k} w_2$, and vice versa;

3. for each child w_1 of v_1 and w_2 of v_2 such that $w_1 \equiv_{\downarrow}^{k} w_2$, $\min(|\bar{w}_1|, k) = \min(|\bar{w}_2|, k)$, where, for $i = 1, 2$, $\bar{w}_i$ is the set of all siblings of w_i (including w_i itself) that are downward k-equivalent to w_i.[6]

For $k = 1$, the third condition above is trivially satisfied. In the literature, downward 1-equivalence is usually referred to as *bisimilarity* [28].

Example 4.4. Consider again the document in Fig. 1. Notice that $v_2 \equiv_{\downarrow}^{k} v_{10}$ for any value of $k \geq 1$. We also have that $v_2 \equiv_{\downarrow}^{1} v_3$, and, for any value of $k \geq 2$, $v_2 \not\equiv_{\downarrow}^{k} v_3$. Finally, notice that $v_3 \not\equiv_{\downarrow}^{k} v_4$ for any value of $k \geq 1$.

The following is immediate from the second condition in the Definition 4.3.

Proposition 4.5. *Let $D = (V, Ed, r, \lambda)$ be a document, let $v_1, v_2 \in V$, and let $k \geq 1$. If $v_1 \equiv_{\downarrow}^{k} v_2$, then v_1 and v_2 have equal height*[7] *in D.*

The following property of downward-k-equivalence will turn out to be very useful in the sequel.

Proposition 4.6. *Let $D = (V, Ed, r, \lambda)$ be a document, and let $k \geq 1$. Let "$\equiv$" be an equivalence relation on V such that, for all $v_1, v_2 \in V$ with $v_1 \equiv v_2$,*

1. *$\lambda(v_1) = \lambda(v_2)$;*
2. *for each child w_1 of v_1, there exists a child w_2 of v_2 such that $w_1 \equiv w_2$, and vice versa; and*
3. *for each child w_1 of v_1 and each child w_2 of v_2 such that $w_1 \equiv w_2$, $\min(|\tilde{w}_1|, k) = \min(|\tilde{w}_2|, k)$, where, for $i = 1, 2$, $\tilde{w}_i$ is the set of all siblings of w_i (including w_i itself) that are equivalent to w_i under "$\equiv$."*

Then, for all $v_1, v_2 \in V$, $v_1 \equiv v_2$ implies $v_1 \equiv_{\downarrow}^{k} v_2$.

Proof. By induction of the height of v_1.

If v_1 is a leaf, the second condition above implies that v_2 must also be a leaf. By the first condition, $\lambda(v_1) = \lambda(v_2)$. Hence, $v_1 \equiv_{\downarrow}^{k} v_2$.

If v_1 is not a leaf, we still have, by the first condition, that $\lambda(v_1) = \lambda(v_2)$. Hence the first condition in the definition of $v_1 \equiv_{\downarrow}^{k} v_2$ (Definition 4.3) is satisfied.

The second condition in the definition of $v_1 \equiv_{\downarrow}^{k} v_2$ follows from the second condition above and the induction hypothesis.

It remains to show that also the third condition in the definition of $v_1 \equiv_{\downarrow}^{k} v_2$ holds. Thereto, let w_1 be a child of v_1 and w_2 be a child of v_2 such that $w_1 \equiv_{\downarrow}^{k} w_2$. We show that $\min(|\bar{w}_1|, k) = \min(|\bar{w}_2|, k)$, where, for $i = 1, 2$, $\bar{w}_i$ is the set of all siblings of w_i (including w_i itself) that are downward k-equivalent to w_i. Let $\{W_{11}, \dots, W_{1\ell}\}$ be the coarsest partition of $\bar{w}_1$ in $\equiv$-equivalent nodes, and let $\{W_{21}, \dots, W_{2\ell}\}$ be the coarsest partition of $\bar{w}_2$ in $\equiv$-equivalent nodes. By the induction hypothesis and the second condition above, both partitions have indeed the same size. It follows furthermore that no node of $\bar{w}_1$ is $\equiv$-equivalent with a child of v_1 outside $\bar{w}_1$, and that no node of $\bar{w}_2$ is $\equiv$-equivalent with a child of v_2 outside $\bar{w}_2$. Without loss of generality, we may assume that, for $i = 1, \dots, \ell$, every node in W_{1i} is $\equiv$-equivalent to every node in W_{2i}. Hence, by the third condition above, $\min(|W_{1i}|, k) = \min(|W_{2i}|, k)$. We now distinguish two cases.

1. For $i = 1, \dots, \ell$, $|W_{1i}| < k$. Then, for $i = 1, \dots, \ell$, $|W_{1i}| = |W_{2i}|$. It follows that $|\bar{w}_1| = |\bar{w}_2|$, and, hence, also that $\min(|\bar{w}_1|, k) = \min(|\bar{w}_2|, k)$.
2. For some i, $1 \leq i \leq \ell$, $|W_{1i}| \geq k$. Then, $|W_{2i}| = |W_{1i}| \geq k$. Hence, $|\bar{w}_1| \geq k$ and $|\bar{w}_2| \geq k$. It follows that $\min(|\bar{w}_1|, k) = \min(|\bar{w}_2|, k) = k$.

We conclude that, in both cases, the third condition in the definition of $v_1 \equiv_{\downarrow}^{k} v_2$ is also satisfied. □

So, given a document $D = (V, Ed, r, \lambda)$, downward-k-equivalence is the coarsest equivalence relation on V satisfying Proposition 4.6.

A straightforward application of Proposition 4.6 yields

Corollary 4.7. *Let $k \geq 1$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. If $v_1 \equiv_{\downarrow}^{k+1} v_2$, then $v_1 \equiv_{\downarrow}^{k} v_2$.*

[6] For a set A, $|A|$ denotes the cardinality of A.

[7] By the height of a node, we mean the length of the longest path from that node to a leaf.

Proof. It suffices to observe that "$\equiv_{\downarrow}^{k+1}$" is an equivalence relation satisfying Proposition 4.6 for the value of k in the statement of the Corollary, above. For the first two conditions in Proposition 4.6, this follows immediately from the corresponding conditions in Definition 4.3. For the third condition in Proposition 4.6, this also follows from the third condition in Definition 4.3 if one takes into account that, for arbitrary sets A and B, $\min(|A|, k+1) = \min(|B|, k+1)$ implies that $\min(|A|, k) = \min(|B|, k)$. $\square$

4.2.2. Upward distinguishability

If we only look upward in the document, there is but one reasonable definition of node distinguishability, as each node has at most one parent. In contrast with the downward case, the recursion in it is on the *depth* of the first node.

Definition 4.8. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. Then, v_1 and v_2 are *upward-equivalent*, denoted $v_1 \equiv_{\uparrow} v_2$, if

1. $\lambda(v_1) = \lambda(v_2)$;
2. v_1 is the root if and only if v_2 is the root; and
3. if v_1 and v_2 are not the root, and u_1 and u_2 are the parents of v_1 and v_2, respectively, then $u_1 \equiv_{\uparrow} u_2$.

It is easily seen that two nodes are upward-equivalent if the paths from the root to these two nodes are isomorphic in the sense that they have the same length and corresponding nodes have the same label.

Example 4.9. In the example document of Fig. 1 we have, e.g., that $v_6 \equiv_{\uparrow} v_7$, $v_8 \equiv_{\uparrow} v_9$, $v_{11} \equiv_{\uparrow} v_{12}$, but $v_8 \not\equiv_{\uparrow} v_{13}$.

4.2.3. Two-way distinguishability

If we look both upward and downward in a document, we can define a notion of equivalence by combining the definitions of upward- and k-downward-equivalence: two nodes are k-equivalent if they are upward-equivalent, and if corresponding nodes on the isomorphic paths from the root to these nodes are k-downward-equivalent. More formally, we have the following recursive definition, where the recursion is on the depth of the first node.

Definition 4.10. Let $D = (V, Ed, r, \lambda)$ be a document, let $v_1, v_2 \in V$, and let $k \geq 1$. Then, v_1 and v_2 are *k-equivalent*, denoted $v_1 \equiv_{\updownarrow}^{k} v_2$, if

1. $v_1 \equiv_{\downarrow}^{k} v_2$;
2. v_1 is the root if and only if v_2 is the root; and
3. if v_1 and v_2 are not the root, and u_1 and u_2 are the parents of v_1 and v_2, respectively, then $u_1 \equiv_{\updownarrow}^{k} u_2$.

Stated in a nonrecursive way, two nodes are k-equivalent if the paths from the root to these two nodes have equal length and corresponding nodes on these two paths are downward-k-equivalent.

Example 4.11. Consider again the document in Fig. 1. We have that, e.g., $v_5 \equiv_{\updownarrow}^{1} v_6 \equiv_{\updownarrow}^{1} v_7$, but no two of these nodes are k-equivalent for any value of $k \geq 2$. Also, $v_5 \not\equiv_{\updownarrow}^{k} v_8$ and $v_8 \not\equiv_{\updownarrow}^{k} v_{13}$, for any value of $k \geq 1$.

By a straightforward inductive argument, the following is immediate from Corollary 4.7.

Proposition 4.12. *Let $k \geq 1$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. If $v_1 \equiv_{\updownarrow}^{k+1} v_2$, then $v_1 \equiv_{\updownarrow}^{k} v_2$.*

4.3. Distinguishability of pairs of nodes at the syntactic level

We also define notions of distinguishability of *pairs* of nodes, by requiring that the pairs have subsumed or congruent signatures and that corresponding nodes on the (undirected) paths between begin and end points of both pairs are related under one of the notions defined in Subsection 4.2.

Definition 4.13. Let $D = (V, Ed, r, \lambda)$ be a document, let ϑ be one of the syntactic relationships between nodes defined in Subsection 4.2, and let $v_1, w_1, v_2, w_2 \in V$. Then, (v_1, w_1) *ϑ-subsumes* (v_2, w_2), denoted $(v_1, w_1) \gtrsim_{\vartheta} (v_2, w_2)$ (respectively, (v_1, w_1) and (v_2, w_2) are *ϑ-congruent*, denoted $(v_1, w_1) \cong_{\vartheta} (v_2, w_2)$) if

Table 3
Distinguishability notions of Section 4.

Distinguishability Notion	Notation	Defined in
Expression-related	$\geq_{\mathrm{exp}}$	Definition 4.1
Expression-equivalent	$\equiv_{\mathrm{exp}}$	Definition 4.1
Downward-k-equivalent	$\equiv^k_{\downarrow}$	Definition 4.3
Upward-equivalent	$\equiv_{\uparrow}$	Definition 4.8
k-equivalent	$\equiv^k_{\updownarrow}$	Definition 4.10
ϑ-subsumes	$\gtrsim_{\vartheta}$	Definition 4.13
ϑ-congruent	$\cong_{\vartheta}$	Definition 4.13

1. $(v_1, w_1) \gtrsim (v_2, w_2)$ (respectively, $(v_1, w_1) \cong (v_2, w_2)$); and
2. for each node y_1 on the path form v_1 to w_1, $y_1 \vartheta y_2$, where y_2 is the unique ancestor of v_2 or w_2 or both for which $(v_2, y_2) \in \mathrm{sig}(v_1, y_1)(D)$ (or, equivalently, $(y_2, w_2) \in \mathrm{sig}(y_1, w_1)(D)$).[8]

Example 4.14. Consider again the document in Fig. 1. We have that, e.g., $(v_2, v_5) \cong_{\equiv^k_{\downarrow}} (v_3, v_6)$ for $k = 1$ but not for any higher value of k; $(v_2, v_5) \cong_{\equiv^k_{\downarrow}} (v_{10}, v_{13})$ for any value of $k \geq 1$; $(v_2, v_5) \cong_{\equiv_{\uparrow}} (v_4, v_9)$; $(v_5, v_6) \cong_{\equiv^k_{\updownarrow}} (v_5, v_7)$ for any value of $k \geq 1$; and $(v_6, v_7) \gtrsim_{\equiv^1_{\updownarrow}} (v_2, v_5)$, but not the other way around.

From Proposition 3.4, (1), the following is obvious.

Proposition 4.15. *Let $D = (V, Ed, r, \lambda)$ be a document, let ϑ be one of the syntactic relationships between nodes defined in Subsection 4.2, and let $v_1, v_2, w_1, w_2 \in V$. If v_1 is an ancestor of w_1 or vice versa, $(v_1, w_1) \cong_{\vartheta} (v_2, w_2)$ if and only if $(v_1, w_1) \gtrsim_{\vartheta} (v_2, w_2)$.*

Finally, from Definitions 4.10 and 4.13, the following is also obvious.

Proposition 4.16. *Let $D = (V, Ed, r, \lambda)$ be a document, let $v_1, v_2 \in V$, and let $k \geq 1$. Then, $v_1 \equiv^k_{\updownarrow} v_2$ if and only if $(r, v_1) \cong_{\equiv^k_{\downarrow}} (r, v_2)$.*

Table 3 summarizes all of the distinguishability notions presented in this section. The balance of the paper is devoted to identifying the languages which correspond in expressive power to each of these notions.

5. Strictly downward languages

We call a language *downward* if, for any expression e and for any node v of the given document D, all nodes in $e(D)(v)$ are descendants of v. In this section, we consider languages with the stronger property that $e(D)(v) = e(D')(v)$, where D' is the subtree of D rooted at v. We shall call such languages *strictly downward*. Downward languages that are *not* strictly downward will be called *weakly downward* and are the subject of Section 6.

Considering the nonbasic operations in Table 1, the language $\mathcal{X}(E)$ is strictly downward if and only if E does *not* contain upward navigation ("$\uparrow$"), second projection ("π_2"), and inverse ("$.^{-1}$"). It is the purpose of this section to investigate the expressive power of these languages at the document level, and, in some cases, derive actual characterizations.

5.1. Sufficient conditions for expression equivalence

If e is an expression in a downward language $\mathcal{X}(E)$, then it follows immediately from the definition that, given a node v of the document D under consideration, each node in $e(D)(v)$ is a descendant of v. Therefore, we only need to consider ancestor-descendant pairs of nodes, for which corresponding notions of subsumption and congruence coincide (Proposition 4.15).

The following property of $\equiv^k_{\downarrow}$-congruence, $k \geq 1$, for ancestor-descendant pairs of nodes will turn out to be very useful.

Lemma 5.1. *Let $D = (V, Ed, r, \lambda)$ be a document, let $v_1, w_1, v_2 \in V$ such that w_1 is a descendant of v_1, and let $k \geq 1$. If $v_1 \equiv^k_{\downarrow} v_2$, then v_2 has a descendant w_2 such that $(v_1, w_1) \cong_{\equiv^k_{\downarrow}} (v_2, w_2)$.*

Proof. The proof is by induction of the length of the path from v_1 to w_1. If $w_1 = v_1$, then, obviously, Lemma 5.1 is satisfied for $w_2 := v_2$. If $w_1 \neq v_1$, then let y_1 be the child of v_1 on the path to w_1. By Definition 4.3, v_2 has a child y_2

[8] In the sequel, we call y_1 and y_2 *corresponding* nodes.

such that $y_1 \equiv_\downarrow^k y_2$. By the induction hypothesis, y_2 has a descendant w_2 in D such that $(y_1, w_1) \cong_{\equiv_\downarrow^k} (y_2, w_2)$. Obviously, $(v_1, w_1) \cong_{\equiv_\downarrow^k} (v_2, w_2)$. □

We now link $\equiv_\downarrow^k$-congruence of ancestor-descendant pairs of nodes with expressibility in strictly downward languages.

Proposition 5.2. *Let $k \geq 1$, and let E be the set of all nonbasic operations in Table 1, except for upward navigation ("$\uparrow$"), second projection ("π_2"), inverse ("$.^{-1}$"), and selection on at least m children satisfying some condition ("$\mathrm{ch}_{\geq m}(.)$") for $m > k$. Let e be an expression in $\mathcal{X}(E)$. Let $D = (V, Ed, r, \lambda)$ be a document, let $v_1, w_1, v_2, w_2 \in V$ such that, for $i = 1, 2$, w_i is a descendant of v_i, and $(v_1, w_1) \cong_{\equiv_\downarrow^k} (v_2, w_2)$. Then, $(v_1, w_1) \in e(D)$ if and only if $(v_2, w_2) \in e(D)$.*

Proof. By symmetry, it suffices to prove that $(v_1, w_1) \in e(D)$ implies $(v_2, w_2) \in e(D)$. We use structural induction. As the base case for the atomic operators $\emptyset$, ε, $\hat{\ell}$ ($\ell \in \mathcal{L}$), and $\downarrow$ is straightforward, we focus on the induction step.

1. $e := e_1/e_2$, with e_1 and e_2 satisfying Proposition 5.2. Assume that $(v_1, w_1) \in e(D)$. Then there exists $y_1 \in V$ such that $(v_1, y_1) \in e_1(D)$ and $(y_1, w_1) \in e_2(D)$. By the strictly downward nature of $\mathcal{X}(E)$, y_1 is on the path from v_1 to w_1. Let y_2 be the node on the path from v_2 to w_2 corresponding to y_1. Obviously, $(v_1, y_1) \equiv_\downarrow^k (v_2, y_2)$ and $(y_1, w_1) \equiv_\downarrow^k (y_2, w_2)$. By the induction hypothesis, $(v_2, y_2) \in e_1(D)$ and $(y_2, w_2) \in e_2(D)$. Hence, $(v_2, w_2) \in e(D)$.
2. $e := \pi_1(f)$, with f satisfying Proposition 5.2. Assume that $(v_1, w_1) \in e(D)$. Then, necessarily $v_1 = w_1$, and, consequently, $v_2 = w_2$. From $(v_1, v_1) \in \pi_1(f)(D)$, it follows that there exists $z_1 \in V$ such that $(v_1, z_1) \in f(D)$. Since $v_1 \equiv_\downarrow^k v_2$, it also follows, by Lemma 5.1, that there exists a descendant z_2 of w_2 such that $(v_1, z_1) \cong_{\equiv_\downarrow^k} (v_2, z_2)$. By the induction hypothesis, $(v_2, z_2) \in f(D)$. Hence, $(v_2, v_2) \in e(D)$.
3. $e := \mathrm{ch}_{\geq m}(f)$, with $m \leq k$ and f satisfying Proposition 5.2. Assume that $(v_1, w_1) \in \mathrm{ch}_{\geq m}(f)(D)$. Hence, $v_1 = w_1$, which in turn implies $v_2 = w_2$. Let $\downarrow/\pi_1(f)(D)(v_1) = Y_1$ and let $\downarrow/\pi_1(f)(D)(v_2) = Y_2$. By assumption, $|Y_1| \geq m$. Now, let y be a child of v_1 in Y_1 or a child of v_2 in Y_2, and let z be a child of v_1 not in Y_1 or a child of v_2 not in Y_2. By assumption, there exists $y' \in V$ such that $(y, y') \in f(D)$. Now, suppose that $y \equiv_\downarrow^k z$. Then, by Proposition 5.1, there exists $z' \in V$ such that $(y, y') \cong_{\equiv_\downarrow^k} (z, z')$. But then, by the induction hypothesis, $(z, z') \in f(D)$, contrary to our assumptions. We may therefore conclude that $y \not\equiv_\downarrow^k z$. Since furthermore $v_1 \equiv_\downarrow^k v_2$, it follows that, for all $y_1 \in Y_1$, there exists $y_2 \in Y_2$ such that $y_1 \equiv_\downarrow^k y_2$, and vice versa. Hence, for some $n \geq 1$, we can write $Y_1 = Y_{11} \cup \ldots \cup Y_{1n}$ and $Y_2 = Y_{21} \cup \ldots \cup Y_{2n}$ such that
   (a) $Y_{11}, \ldots, Y_{1n}$ are maximal sets of mutually downward-k-equivalent children of v_1, and are hence pairwise disjoint;
   (b) $Y_{21}, \ldots, Y_{2n}$ are maximal sets of mutually downward-k-equivalent children of v_2, and are hence pairwise disjoint; and
   (c) for $i = 1, \ldots, n$, each node of Y_{1i} is downward-k-equivalent to each node of Y_{2i}.
   If, for some i, $|Y_{1i}| \geq k$, it follows from $v_1 \equiv_\downarrow^k v_2$ that $|Y_{2i}| \geq k$, and, hence, that $|Y_2| \geq k \geq m$. If, on the other hand, for $i = 1, \ldots, n$, $|Y_{1i}| < k$, it follows from $v_1 \equiv_\downarrow^k v_2$ that $|Y_{1i}| = |Y_{2i}|$, and, hence, that $|Y_1| = |Y_2|$. Since $|Y_1| \geq m$, it follows that, also in this case, $|Y_2| \geq m$. We may thus conclude that, in all cases, $|Y_2| \geq m$, and, hence, that $(v_2, v_2) \in \mathrm{ch}_{\geq m}(f)(D) = e(D)$.
4. $e := e_1 \cup e_2$, with e_1 and e_2 satisfying Proposition 5.2. Assume that $(v_1, w_1) \in e(D)$. Then, $(v_1, w_1) \in e_1(D)$ or $(v_1, w_1) \in e_2(D)$. Without loss of generality, assume the former. Then, by the induction hypothesis, $(v_2, w_2) \in e_1(D)$. Hence, $(v_2, w_2) \in e(D)$.
5. $e := e_1 \cap e_2$, with e_1 and e_2 satisfying Proposition 5.2. Assume that $(v_1, w_1) \in e(D)$. Then, $(v_1, w_1) \in e_1(D)$ and $(v_1, w_1) \in e_2(D)$. It follows by the induction hypothesis that $(v_2, w_2) \in e_1(D)$ and $(v_2, w_2) \in e_2(D)$. Hence, $(v_2, w_2) \in e(D)$.
6. $e := e_1 - e_2$, with e_1 and e_2 satisfying Proposition 5.2. Assume that $(v_1, w_1) \in e(D)$. Then $(v_1, w_1) \in e_1(D)$ and $(v_1, w_1) \notin e_2(D)$. By the induction hypothesis, $(v_2, w_2) \in e_1(D)$ and $(v_2, w_2) \notin e_2(D)$. (Indeed, if $(v_2, w_2) \in e_2(D)$, then, again by the induction hypothesis, $(v_1, w_1) \in e_2(D)$, a contradiction.) Hence, $(v_2, w_2) \in e(D)$. □

Corollary 5.3. *Let $k \geq 1$, and let E be the set of all nonbasic operations in Table 1, except for upward navigation ("$\uparrow$"), second projection ("π_2"), inverse ("$.^{-1}$"), and selection on at least m children ("$\mathrm{ch}_{\geq m}(.)$") for $m > k$. Let e be an expression in $\mathcal{X}(E)$. Let $D = (V, Ed, r, \lambda)$ be a document, let $v_1, v_2 \in V$ such that $v_1 \equiv_\downarrow^k v_2$, and let w_1 be a descendant of v_1. If $(v_1, w_1) \in e(D)$, then there exists a descendant w_2 of v_2 such that $(v_2, w_2) \in e(D)$.*

Proof. By Lemma 5.1, there exists a descendant w_2 of v_2 such that $(v_1, w_1) \cong_{\equiv_\downarrow^k} (v_2, w_2)$. By Proposition 5.2, it now follows that $(v_2, w_2) \in e(D)$. □

Corollary 5.4. *Let $k \geq 1$, and let E be a set of nonbasic operations in Table 1 not containing upward navigation ("$\uparrow$"), second projection ("π_2"), inverse ("$.^{-1}$"), or selection on at least m children satisfying some condition ("$\mathrm{ch}_{\geq m}(.)$") for $m > k$. Consider the language $\mathcal{X}(E)$ or $\mathcal{C}(E)$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. If $v_1 \equiv_\downarrow^k v_2$, then $v_1 \equiv_{\exp} v_2$.*

Proof. Let e be an expression such that $e(D)(v_1) \neq \emptyset$. Hence, there exists a descendant w_1 of v_1 such that $(v_1, w_1) \in e(D)$. By Corollary 5.3, there exists a descendant w_2 of v_2 such that $(v_2, w_2) \in e(D)$, so $e(D)(v_2) \neq \emptyset$. By symmetry, the converse also holds. Hence, $v_1 \equiv_{\text{exp}} v_2$. □

Hence, downward-k-equivalence is a sufficient condition for expression-equivalence under a strictly downward language if $\text{ch}_{\geq m}(.)$ cannot be expressed for $m > k$. This restriction cannot be removed, as shown by this counterexample:

Example 5.5. Consider again the document in Fig. 1. We established in Example 4.4 that $v_2 \equiv_{\downarrow}^{1} v_3$, but $v_2 \not\equiv_{\downarrow}^{2} v_3$. In the language $\mathcal{X}(\text{ch}_{\geq 2}(.))$, clearly $v_2 \not\equiv_{\text{exp}} v_3$, as $\text{ch}_{\geq 2}(\varepsilon)(D)(v_2) = \emptyset$, while $\text{ch}_{\geq 2}(\varepsilon)(D)(v_3) \neq \emptyset$.

5.2. Necessary conditions for expression equivalence

We now explore requirements on the set of nonbasic operations expressible in a language under which downward-k-equivalence ($k \geq 1$) is necessary for expression-equivalence. As we tried to make as few assumptions as possible, Proposition 5.6 also holds for a class of languages that are *not* (strictly) downward.

Proposition 5.6. *Let $k \geq 1$, and let E be a set of nonbasic operations containing set difference ("$-$"). Consider the language $\mathcal{X}(E)$ or $\mathcal{C}(E)$. Assume that first projection ("π_1") is expressible, as well as selection on at least m children satisfying some condition ("$\text{ch}_{\geq m}(.)$"), for $m = 1, \ldots, k$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. If $v_1 \equiv_{\text{exp}} v_2$, then $v_1 \equiv_{\downarrow}^{k} v_2$.*

Proof. Without loss of generality, we may assume that the language under consideration is $\mathcal{C}(E)$. To prove the result, it suffices to show that expression-equivalence ("$\equiv_{\text{exp}}$") satisfies the conditions of Proposition 4.6. Thus, let $v_1, v_2 \in V$ be such that $v_1 \equiv_{\downarrow}^{k} v_2$.

1. Since $\widehat{\lambda(v_1)}(D)(v_1) \neq \emptyset$, $\widehat{\lambda(v_1)}(D)(v_2) \neq \emptyset$, and hence $\lambda(v_1) = \lambda(v_2)$.
2. Assume that v_1 has a child w_1. Since $\text{ch}_{\geq 1}(\varepsilon)(D)(v_1) \neq \emptyset$, $ch_1(\varepsilon)(D)(v_2) \neq \emptyset$. Hence v_2 has at least one child. Let $w_2^1, \ldots, w_2^n$ be all children of v_2. Suppose for the sake of contradiction that, for $i = 1, \ldots, n$, $w_1 \not\equiv_{\text{exp}} w_2^i$. Then, by Proposition 4.2, there exists an expression e_i in $\mathcal{C}(E)$ such that $e_i(D)(w_1) \neq \emptyset$ and $e_i(D)(w_2^i) = \emptyset$, for $i = 1, \ldots, n$. Now, let $e := \pi_1(e_1)/\ldots/\pi_1(e_n)$. Then, $\text{ch}_{\geq 1}(e)(D)(v_1) \neq \emptyset$ while $\text{ch}_{\geq 1}(e)(D)(v_2) = \emptyset$, a contradiction. Hence, v_2 has a child w_2 such that $w_1 \equiv_{\text{exp}} w_2$. Of course, the same also goes with the roles of v_1 and v_2 reversed.
3. Assume that v_1 has a child w_1 and v_2 has a child w_2 such that $w_1 \equiv_{\text{exp}} w_2$. For $i = 1, 2$, let $\tilde{w}_i$ be the set of all siblings of w_i (including w_i itself) that are expression-equivalent to w_i. As in the previous item, we can construct an expression e in $\mathcal{C}(E)$ such that $e(D)(w_1) \neq \emptyset$ (and hence $e(D)(w) \neq \emptyset$ for each node w in $\tilde{w}_1$ or $\tilde{w}_2$) and $e(D)(w) = \emptyset$ for each sibling of w_1 not in $\tilde{w}_1$ and for each sibling of w_2 not in $\tilde{w}_2$. For the sake of contradiction, assume that $\min(|\tilde{w}_1|, k) \neq \min(|\tilde{w}_2|, k)$. Without loss of generality, assume that $\min(|\tilde{w}_1|, k) < \min(|\tilde{w}_2|, k)$. Hence, $\min(|\tilde{w}_1|, k) = |\tilde{w}_1|$. Let $m := \min(|\tilde{w}_2|, k)$. Then, $\text{ch}_{\geq m}(e)(D)(v_1) = \emptyset$, while $\text{ch}_{\geq m}(e)(D)(v_2) \neq \emptyset$, a contradiction. Hence, $\min(|\tilde{w}_1|, k) = \min(|\tilde{w}_2|, k)$. □

Hence, downward-k-equivalence is a necessary condition for expression-equivalence under a strictly downward language containing first projection ("π_1") and set difference ("$-$") if selection on at least m children satisfying some condition ("$\text{ch}_{\geq m}(.)$") for $m = 1, \ldots, k$ can be expressed.

5.3. Characterization of expression equivalence

We call $\mathcal{X}(\downarrow, \pi_1, \text{ch}_{\geq 1}(.), \ldots, \text{ch}_{\geq k}(.), -)$ ($\mathcal{C}(\downarrow, \pi_1, \text{ch}_{\geq 1}(.), \ldots, \text{ch}_{\geq k}(.), -)$) the *strictly downward* (*core*) *XPath algebra with counting up to k*. As these are the languages under consideration which effectively contain downward navigation ("$\downarrow$") and satisfy both Corollary 5.4 and Proposition 5.6, we obtain

Theorem 5.7. *Let $k \geq 1$, and consider the strictly downward (core) XPath algebra with counting up to k. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. Then $v_1 \equiv_{\text{exp}} v_2$ if and only if $v_1 \equiv_{\downarrow}^{k} v_2$.*

A special case arises when $k = 1$, since, by Proposition 2.4, selection on at least one child satisfying some condition ("$\text{ch}_{\geq 1}(.)$") can be expressed in terms of the other operations required by Theorem 5.7. We call $\mathcal{X}(\downarrow, \pi_1, -)$ ($\mathcal{C}(\downarrow, \pi_1, -)$) the *strictly downward* (*core*) *XPath algebra*. We have the following.

Corollary 5.8. *Consider the strictly downward (core) XPath algebra. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. Then $v_1 \equiv_{\text{exp}} v_2$, if and only if $v_1 \equiv_{\downarrow}^{1} v_2$.*

5.4. Characterization of navigational expressiveness

We now investigate the expressiveness of strictly downward languages at the document level. Given a document, we try to characterize when a set of pairs of nodes of that document is the result of some query in the language under consideration applied to that document. Such results are often referred to as BP-characterizations, after Bancilhon [25] and Paredaens [26] who first proved such results for Codd's relational calculus and algebra, respectively (cf. [27]).

We start by proving a converse to Proposition 5.2.

Proposition 5.9. *Let $k \geq 1$, and let E be a set of nonbasic operations containing downward navigation ("$\downarrow$") and set difference ("$-$"). Consider the language $\mathcal{X}(E)$ or $\mathcal{C}(E)$. Assume that first projection ("π_1") is expressible, as well as selection on at least m children satisfying some condition ("$\mathrm{ch}_{\geq m}(.)$"), for $m = 1, \ldots, k$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, w_1, v_2, w_2 \in V$ be such that, for $i = 1, 2$, w_i is a descendant of v_i, and, for each expression e, $(v_1, w_1) \in e(D)$ if and only if $(v_2, w_2) \in e(D)$. Then $(v_1, w_1) \cong_{\equiv_\downarrow^k} (v_2, w_2)$.*

Proof. By assumption, $(v_2, w_2) \in \mathrm{sig}(v_1, w_1)(D)$, and vice versa. Hence, $(v_1, w_1) \cong (v_2, w_2)$. Now, let y_1 be any node on the path from v_1 to w_1, and let y_2 be the corresponding node on the path from v_2 to w_2. By construction, $(v_1, y_1) \cong (v_1, y_2)$ and $(y_1, w_1) \cong (y_2, w_2)$. It remains to show that $y_1 \equiv_\downarrow^k y_2$. Thereto, let f be any expression in the language such that $f(D)(y_1) \neq \emptyset$. Let $e := \mathrm{sig}(v_1, y_1)/\pi_1(f)/\mathrm{sig}(y_1, w_1)$. By construction, $(v_1, w_1) \in e(D)$, so $(v_2, w_2) \in e(D)$, which implies $f(D)(y_2) \neq \emptyset$. Since the same also holds vice versa, it follows that $y_1 \equiv_{\mathrm{exp}} y_2$. The desired result now follows from Proposition 5.6. □

Combining Propositions 5.2 and 5.9, we obtain the following.

Corollary 5.10. *Let $k \geq 1$, and consider the strictly downward (core) XPath algebra with counting up to k. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, w_1, v_2, w_2 \in V$ be such that, for $i = 1, 2$, w_i is a descendant of v_i. Then, the property that, for each expression e, $(v_1, w_1) \in e(D)$ if and only if $(v_2, w_2) \in e(D)$, is equivalent to $(v_1, w_1) \cong_{\equiv_\downarrow^k} (v_2, w_2)$.*

In order to state our first BP-result, we need the following two lemmas.

Lemma 5.11. *Let $k \geq 1$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1 \in V$. There exists an expression e_{v_1} in the strictly downward core XPath algebra with counting up to k such that, for each $v_2 \in V$, $e_{v_1}(D)(v_2) \neq \emptyset$ if and only if $v_1 \equiv_\downarrow^k v_2$.*

Proof. Let $w \in V$ be such that $v_1 \not\equiv_\downarrow^k w$. By Theorem 5.7, $v_1 \not\equiv_{\mathrm{exp}} w$. By Proposition 4.2, there exists an expression $f_{v_1,w}$ in the strictly downward core XPath algebra with counting up to k such that $f_{v_1,w}(D)(v_1) \neq \emptyset$ and $f_{v_1,w}(D)(w) = \emptyset$. Now, let e_{v_1} be the composition of the expressions $\pi_1(f_{v_1,w})$ for all $w \in V$ for which $v_1 \not\equiv_\downarrow^k w$. By construction, $e_{v_1}(D)(v_1) \neq \emptyset$. Now consider $v_2 \in V$. If $v_1 \equiv_\downarrow^k v_2$, then, by Theorem 5.7, $v_1 \equiv_{\mathrm{exp}} v_2$, and, hence, $e_{v_1}(D)(v_2) \neq \emptyset$. If $v_1 \not\equiv_\downarrow^k v_2$, then, by construction, $e_{v_1}(D)(v_2) = \emptyset$. □

Lemma 5.12. *Let $k \geq 1$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, w_1 \in V$ be such that w_1 is a descendant of v_1. There exists an expression e_{v_1,w_1} in the strictly downward core XPath algebra with counting up to k such that, for all $v_2, w_2 \in V$ with w_2 a descendant of v_2, $(v_2, w_2) \in e_{v_1,w_1}(D)$ if and only if $(v_1, w_1) \cong_{\equiv_\downarrow^k} (v_2, w_2)$.*

Proof. Let $y_1 \in V$. By Lemma 5.11, there exists an expression e_{y_1} in the strictly downward core XPath algebra with counting up to k such that, for all $y_2 \in V$, $e_{y_1}(D)(y_2) \neq \emptyset$ if and only if $y_1 \equiv_\downarrow^k y_2$. Now, let $v_1, w_1 \in V$ be such that w_1 is a descendant of v_1, and let $v_1 = y_{11}, \ldots, y_{1n} = w_1$ be the path from v_1 to w_1 in D. Let $e_{v_1,w_1} := \pi_1(e_{y_{11}})/\downarrow/\pi_1(e_{y_{12}})/\ldots\downarrow/\pi_1(e_{y_{1n}})$. By construction, $(v_1, w_1) \in e_{v_1,w_1}(D)$. Let $v_2, w_2 \in V$ such that w_2 is a descendant of v_2. If $(v_1, w_1) \cong_{\equiv_\downarrow^k} (v_2, w_2)$, then, by Corollary 5.10, $(v_2, w_2) \in e_{v_1,w_1}(D)$. Conversely, if $(v_2, w_2) \in e_{v_1,w_1}(D)$, then, by construction, $(v_1, w_1) \cong (v_2, w_2)$. Thus, let $v_2 = y_{21}, \ldots, y_{2n} = w_2$ be the path from v_2 to w_2 in D. Again by construction, it follows that, for $j = 1, \ldots, n$, $e_{y_{1j}}(D)(y_{2j}) \neq \emptyset$, or, equivalently, that $y_{1j} \equiv_\downarrow^k y_{2j}$. Hence, $(v_1, w_1) \equiv_\downarrow^k (v_2, w_2)$. □

We are now ready to state the actual result.

Theorem 5.13. *Let $k \geq 1$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $R \subseteq V \times V$. Then, there exists an expression e in the strictly downward (core) XPath algebra with counting up to k such that $e(D) = R$ if and only if,*

1. *for all $v, w \in V$, $(v, w) \in R$ implies w is a descendant of v; and,*
2. *for all $v_1, w_1, v_2, w_2 \in V$ such that, for $i = 1, 2$, w_i is a descendant of v_i and $(v_1, w_1) \cong_{\equiv^k_\downarrow} (v_2, w_2)$, $(v_1, w_1) \in R$ implies $(v_2, w_2) \in R$.*

Proof. To see the "only if," it suffices to notice that the first condition follows from the downward character of the language, and the second from Corollary 5.10. The remainder of the proof concerns the "if." Let $v_1, w_1 \in V$ such that w_1 is a descendant of v_1. By Lemma 5.12, there exists an expression e_{v_1,w_1} in $\mathcal{C}(E)$ such that, for all $v_2, w_2 \in V$, $(v_2, w_2) \in e_{v_1,w_1}(D)$ if and only if $(v_1, w_1) \cong_{\equiv^k_\downarrow} (v_2, w_2)$. Let $e := \bigcup_{(v_1,w_1) \in R} e_{v_1,w_1}$. Clearly, $R \subseteq e(D)$. It remains to show that $e(D) \subseteq R$. Thus, let $v_2, w_2 \in V$ be such that $(v_2, w_2) \in e(D)$. By construction, there exist $v_1, w_1 \in V$ such that w_1 is a descendant of v_1 and $(v_2, w_2) \in e_{v_1,w_1}(D)$. Hence, $(v_1, w_1) \cong_{\equiv^k_\downarrow} (v_2, w_2)$. But then, by assumption, also $(v_2, w_2) \in R$. □

We specialize Theorem 5.13 to the strictly downward (core) XPath algebra.

Corollary 5.14. *Let $D = (V, Ed, r, \lambda)$ be a document, and let $R \subseteq V \times V$. There exists an expression e in the strictly downward (core) XPath algebra such that $e(D) = R$ if and only if,*

1. *for all $v, w \in V$, $(v, w) \in R$ implies w is a descendant of v; and*
2. *for all $v_1, w_1, v_2, w_2 \in V$ such that, for $i = 1, 2$, w_i is a descendant of v_i and $(v_1, w_1) \equiv^1_\downarrow (v_2, w_2)$, $(v_1, w_1) \in R$ implies $(v_2, w_2) \in R$.*

We next recast Theorem 5.13 in terms of node-level navigation.

Theorem 5.15. *Let $k \geq 1$. Let $D = (V, Ed, r, \lambda)$ be a document, let $v \in V$, and let $W \subseteq V$. Then there exists an expression e in the strictly downward (core) XPath algebra with counting up to k such that $e(D)(v) = W$ if and only if all nodes of W are descendants of v, and, for all $w_1, w_2 \in V$ such that $(v, w_1) \cong_{\equiv^k_\downarrow} (v, w_2)$, $w_1 \in W$ implies $w_2 \in W$.*

Proof. As the "only if" is straightforward, we only consider the "if." Thus, let $W \subseteq V$ satisfy the property that all nodes of W are descendants of v, and, for all $w_1, w_2 \in V$ with $w_1 \equiv^k_\downarrow w_2$, $w_1 \in W$ implies $w_2 \in W$. Let $R := \{(v', w_2) \mid (\exists\, w_1 \in W)((v, w_1) \cong_{\equiv^k_\downarrow} (v', w_2))\}$. Clearly, R satisfies the properties of Theorem 5.13, a straightforward application of which yields the desired result. □

We also specialize Theorem 5.15 to the strictly downward (core) XPath algebra.

Corollary 5.16. *Let $D = (V, Ed, r, \lambda)$ be a document, let $v \in V$, and let $W \subseteq V$. Then there exists an expression e in the strictly downward (core) XPath algebra such that $e(D)(v) = W$ if and only if all nodes of W are descendants of v, and, for all $w_1, w_2 \in V$ such that $(v, w_1) \cong_{\equiv^1_\downarrow} (v, w_2)$, $w_1 \in W$ implies $w_2 \in W$.*

A special case of Theorem 5.15 arises for navigation from the *root* only:

Theorem 5.17. *Let $k \geq 1$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $W \subseteq V$. Then there exists an expression e in the strictly downward (core) XPath algebra with counting up to k such that $e(D)(r) = W$ if and only if, for all $w_1, w_2 \in V$ such that $w_1 \equiv^k_\updownarrow w_2$, $w_1 \in W$ implies $w_2 \in W$.*

Proof. By Theorem 5.15, $e(D)(r) = W$ if and only if, for all $w_1, w_2 \in V$ with $(r, w_1) \cong_{\equiv^k_\downarrow} (r, w_2)$, $w_1 \in W$ implies $w_2 \in W$. By Proposition 4.16, $(r, w_1) \cong_{\equiv^k_\downarrow} (r, w_2)$ is equivalent to $w_1 \equiv^k_\updownarrow w_2$. □

The specialization of Theorem 5.17 to the case of the strictly downward (core) XPath algebra is as follows.

Corollary 5.18. *Let $D = (V, Ed, r, \lambda)$ be a document, and let $W \subseteq V$. Then there exists an expression e in the strictly downward (core) XPath algebra such that $e(D)(r) = W$ if and only if, for all $w_1, w_2 \in V$ such that $w_1 \equiv^1_\updownarrow w_2$, $w_1 \in W$ implies $w_2 \in W$.*

We observe that none of the characterization results above distinguish between the language $\mathcal{X}(E)$ and the corresponding core language $\mathcal{C}(E)$. Actually, for *all* downward languages, the two have the same expressive power, not only at the navigational level for a given document, but also at the level of queries, i.e., for each expression e in $\mathcal{X}(E)$, there exists an equivalent expression e' in $\mathcal{C}(E)$, meaning that, *for each* document D, $e(D) = e'(D)$. To see this, we prove a slightly stronger result.

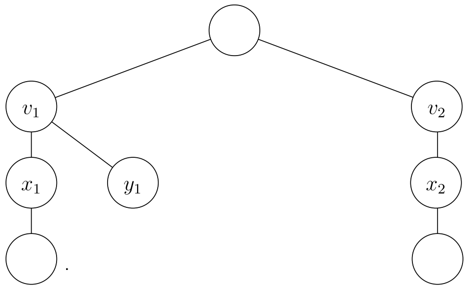


Fig. 3. Document of Example 5.22.

Theorem 5.19. *Let E be a set of nonbasic operations containing downward navigation ("$\downarrow$") and first projection ("π_1"), and not containing upward navigation ("$\uparrow$"), and inverse ("$.^{-1}$"). Let e be an expression in the language under consideration. Except where set difference ("$-$") operations are used as operands in boolean combinations of subexpressions inside a first projection, all intersection ("$\cap$") and set difference operations can be eliminated, to the extent that these operations occur in the language under consideration.*

Proof. The proof goes by structural induction. Therefore, consider the expression $e_1 \cap e_2$, respectively, $e_1 - e_2$ (to the extent these operations occur in the language under consideration), where e_1 and e_2 are expressions not containing eliminable intersection and set difference operations. For $i = 1, 2$, we may write $e_i = c_{i0}/\downarrow/c_{i1}/\downarrow/\ldots/\downarrow/c_{in_{i-1}}/\downarrow/c_{in_i}$, where, for $j = 0, \ldots, n_i$, c_{ij} is an expression in $\mathcal{C}(E)$ with the property that, for each document D, $c_{ij}(D) \subseteq \varepsilon(D)$. We now consider both cases separately.

1. *Intersection*. Clearly, if $n_1 \neq n_2$, then, for each document D, $e_1 \cap e_2(D) = \emptyset = \emptyset(D)$. In the other case, let $n := n_1 = n_2$. For $j = 0, \ldots, n$, let $c_j := c_{1j}/c_{2j}$. Then, $e' := c_0/\downarrow/c_1/\downarrow/\ldots/\downarrow/c_{n-1}/\downarrow/c_n$ is equivalent to $e_1 \cap e_2$.
2. *Difference*. Clearly, if $n_1 \neq n_2$, then, for each document D, $e_1 - e_2(D) = e_1(D)$. In the other case, let $n := n_1 = n_2$. For $j = 0, \ldots, n$, let e'_j be e_1 in which c_{1j} is replaced by $\pi_1(c_{1j} - c_{2j})$, which is an expression of $\mathcal{C}(E)$, equivalent to $c_{1j} - c_{2j}$. Then, $e' = e'_0 \cup e'_1 \cup \ldots \cup e'_{n-1} \cup e'_n$ is equivalent to $e_1 - e_2$. □

Corollary 5.20. *Let E be a set of nonbasic operations containing downward navigation ("$\downarrow$") and first projection ("π_1"), and not containing upward navigation ("$\uparrow$"), and inverse ("$.^{-1}$"). Then, for each expression e in $\mathcal{X}(E)$, there exists an expression e' in $\mathcal{C}(E)$ such that, for each document D, $e(D) = e'(D)$.*

5.5. Strictly downward languages not containing set difference

So far, the characterizations of strictly downward languages involved only languages containing the set difference operator. One could, therefore, wonder if it is possible to provide similar characterizations for languages not containing set difference. However, the absence of set difference and the logical negation that is inherently embedded in it has as a side effect that it is no longer always possible to exploit equivalences or derive them.

5.5.1. Weaker notions of downward and two-way distinguishability

Therefore, one would like to consider an asymmetric version of downward k-equivalence, say "downward k-relatedness," which, for the appropriate language, could correspond to expression relatedness. For $k = 1$, such an approach could lead to the following definitions.

Definition 5.21. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. Then,

1. v_1 and v_2 are *downward-related*, denoted $v_1 \geq_\downarrow v_2$, if
   (a) $\lambda(v_1) = \lambda(v_2)$; and
   (b) for each child w_1 of v_1, there exists a child w_2 of v_2 such that $w_1 \geq_\downarrow w_2$.
2. v_1 and v_2 are *weakly downward-equivalent*, denoted $v_1 \approxeq_\downarrow v_2$, if $v_1 \geq_\downarrow v_2$ and $v_2 \geq_\downarrow v_1$.

Obviously, downward 1-equivalence implies weak downward equivalence. The converse, however, is *not* true, as illustrated by the following, simple example.

Example 5.22. Consider the document in Fig. 3. Labels have been omitted, because they are not relevant in this discussion. (We assume all nodes have the same label.) Obviously, $x_1 \equiv^1_\downarrow x_2$, hence $x_1 \approxeq_\downarrow x_2$. In particular, $x_1 \geq_\downarrow x_2$ and $x_2 \geq_\downarrow x_1$. Also, $y_1 \geq_\downarrow x_2$, as the second condition to be verified is voidly satisfied in this case. We may thus conclude that $v_1 \approxeq_\downarrow v_2$. However, $v_1 \not\equiv^1_\downarrow v_2$, as there is no child of v_2 that is downward 1-equivalent to y_1.

Table 4
Distinguishability notions of Section 5.5.1.

Distinguishability Notion	Notation	Defined in
Downward-related	$\geq_{\downarrow}$	Definition 5.21
Weakly-downward-equivalent	$\approxeq_{\downarrow}$	Definition 5.21
Related	$\geq_{\updownarrow}$	Definition 5.23
Weakly-equivalent	$\approxeq_{\updownarrow}$	Definition 5.23

Notice that, in Example 5.22, there is even no child of v_2 that is *weakly* downward equivalent to y_1! Therefore, we shall not even attempt to generalize Definition 5.21 to the case where $k > 1$, as there is no straightforward way to adapt the third condition of Definition 4.3.

We conclude this digression on alternatives for downward 1-equivalence by providing analogue alternatives for 1-equivalence.

Definition 5.23. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. Then,

1. v_1 and v_2 are *related*, denoted $v_1 \geq_{\updownarrow} v_2$, if
   (a) $v_1 \geq_{\downarrow} v_2$;
   (b) v_1 is the root if and only if v_2 is the root; and
   (c) if v_1 and v_2 are not the root, and u_1 and u_2 are the parents of v_1 and v_2, respectively, then $u_1 \geq_{\updownarrow} u_2$.
2. v_1 and v_2 are *weakly equivalent*, denoted $v_1 \approxeq_{\updownarrow} v_2$, if $v_1 \geq_{\updownarrow} v_2$ and $v_2 \geq_{\updownarrow} v_1$.

Example 5.24. Consider again the document in Fig. 3. Observe that $v_1 \approxeq_{\updownarrow} v_2$. Furthermore, $y_1 \geq_{\updownarrow} x_2$, but not the other way around.

Table 4 summarizes the distinguishability notions presented in this section.

The following analogue of Proposition 4.16 is straightforward.

Proposition 5.25. *Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. Then,*

1. *$v_1 \geq_{\updownarrow} v_2$ if and only if $(r, v_1) \cong_{\geq_{\downarrow}} (r, v_2)$; and*
2. *$v_1 \approxeq_{\updownarrow} v_2$ if and only if $(r, v_1) \cong_{\approxeq_{\downarrow}} (r, v_2)$.*

5.5.2. Characterizing expression equivalence and navigational expressiveness

The approach we shall take here is reviewing the results in Sections 5.1–5.3 and examine to which extent these results in the case where $k = 1$ allow replacing downward 1-equivalence by weak downward equivalence.

We start by observing that the analogue of Lemma 5.1 does not hold. Indeed, in the example document of Example 5.22, shown in Fig. 3, $v_1 \approxeq_{\downarrow} v_2$. Also, there is no child of v_2 that is weakly downward equivalent to x_1. Hence, there is no node z for which $(v_1, x_1) \cong_{\approxeq^1_{\downarrow}} (v_2, z)$. On the other hand, we can restrict Lemma 5.1 to downward relatedness:

Lemma 5.26. *Let $D = (V, Ed, r, \lambda)$ be a document, let $v_1, w_1, v_2 \in V$ such that w_1 is a descendant of v_1. If $v_1 \geq_{\downarrow} v_2$, then v_2 has a descendant w_2 such that $(v_1, w_1) \cong_{\geq_{\downarrow}} (v_2, w_2)$.*

Proposition 5.2 relies on Lemma 5.1 to prove the inductive step for the first projection ("π_1"). It therefore comes as no surprise that we cannot replace downward 1-equivalence by weak downward equivalence, there. Indeed, consider the expression $e := \pi_1(\downarrow/(\varepsilon - \pi_1(\downarrow)))$. In the example document of Example 5.22, shown in Fig. 3, $v_1 \approxeq_{\downarrow} v_2$, and, hence, $(v_1, v_1) \cong_{\approxeq_{\downarrow}} (v_2, v_2)$. Moreover, $(v_1, v_1) \in e(D)$. However, $(v_2, v_2) \notin e(D)$. However, we can "save" Proposition 5.2 by replacing downward 1-equivalence by downward *relatedness*, provided we omit set difference ("$-$") from the set of operations of the language. Indeed, we can then recover the proof, using Lemma 5.26 instead of Lemma 5.1. (Notice that, for the induction step for set difference in the original proof, we must exploit equivalence in both directions to deal with the negation inherent to the difference operation.) In summary, we have the following.

Lemma 5.27. *Let E be the set of all nonbasic operations in Table 1, except for upward navigation ("$\uparrow$"), second projection ("π_2"), inverse ("$.^{-1}$"), selection on at least k children satisfying some condition ("$\mathrm{ch}_{\geq k}(.)$") for $k > 1$, and set difference ("$-$"). Let e be an expression in $\mathcal{X}(E)$. Let $D = (V, Ed, r, \lambda)$ be a document, let $v_1, w_1, v_2, w_2 \in V$ such that, for $i = 1, 2$, w_i is a descendant of v_i, and $(v_1, w_1) \cong_{\geq_{\downarrow}} (v_2, w_2)$. Then, $(v_1, w_1) \in e(D)$ implies $(v_2, w_2) \in e(D)$.*

Two applications of Lemma 5.27 immediately yield the following.

Proposition 5.28. *Let E be the set of all nonbasic operations in Table 1, except for upward navigation ("$\uparrow$"), second projection ("π_2"), inverse ("$.^{-1}$"), selection on at least k children satisfying some condition ("$\text{ch}_{\geq k}(.)$") for $k > 1$, and set difference ("$-$"). Let e be an expression in $\mathcal{X}(E)$. Let $D = (V, Ed, r, \lambda)$ be a document, let $v_1, w_1, v_2, w_2 \in V$ such that, for $i = 1, 2$, w_i is a descendant of v_i, and $(v_1, w_1) \cong_{\approx_\downarrow} (v_2, w_2)$. Then, $(v_1, w_1) \in e(D)$ if and only $(v_2, w_2) \in e(D)$.*

So, Proposition 5.28 is weaker than Proposition 5.2 in the sense that we had to exclude set difference, but stronger in the sense that, in return, we were able to replace the precondition by a weaker one.

The analogues of Corollaries 5.3 and 5.4 are now as follows.

Corollary 5.29. *Let E be the set of all nonbasic operations in Table 1, except for upward navigation ("$\uparrow$"), second projection ("π_2"), inverse ("$.^{-1}$"), selection on at least k children satisfying some condition ("$\text{ch}_{\geq k}(.)$") for $k > 1$, and set difference ("$-$"). Let e be an expression in $\mathcal{X}(E)$. Let $D = (V, Ed, r, \lambda)$ be a document, let $v_1, v_2 \in V$ such that $v_1 \geq_\downarrow v_2$ and let w_1 be a descendant of v_1. If $(v_1, w_1) \in e(D)$, then there exists a descendant w_2 of v_2 such that $(v_2, w_2) \in e(D)$.*

In other words, downward relatedness implies expression relatedness. In a straightforward manner, we can bootstrap this result as follows.

Corollary 5.30. *Let E be a set of nonbasic operations not containing upward navigation ("$\uparrow$"), second projection ("π_2"), inverse ("$.^{-1}$"), selection on at least k children satisfying some condition ("$\text{ch}_{\geq k}(.)$") for $k > 1$, and set difference ("$-$"). Consider the language $\mathcal{X}(E)$ or $\mathcal{C}(E)$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. If $v_1 \approx_\downarrow v_2$, then $v_1 \equiv_{\text{exp}} v_2$.*

We now look to necessary conditions for expression equivalence for strictly downward languages not containing set difference. Notice that the expressibility of set difference is used only once in the proof of Proposition 5.6, namely where Proposition 4.2 is invoked. We do not need this Proposition, however, in the following variation of Proposition 5.6:

Lemma 5.31. *Let E be a set of nonbasic operations containing first projection ("π_1"). Consider the language $\mathcal{X}(E)$ or $\mathcal{C}(E)$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. If $v_1 \geq_{\text{exp}} v_2$, then $v_1 \geq_\downarrow v_2$.*

Two applications of Lemma 5.31 immediately yield the following.

Proposition 5.32. *Let E be a set of nonbasic operations containing first projection ("π_1"). Consider the language $\mathcal{X}(E)$ or $\mathcal{C}(E)$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. If $v_1 \equiv_{\text{exp}} v_2$, then $v_1 \approx_\downarrow v_2$.*

So, Proposition 5.32 is weaker than Proposition 5.6 in the sense that the conclusion is replaced by a weaker one, but stronger in the sense that, in return, we no longer have to rely on the presence of difference.

The languages containing downward navigation ("$\downarrow$") and satisfying both Corollary 5.30 and Proposition 5.32 are $\mathcal{X}(\downarrow, \pi_1, \cap)$ and $\mathcal{C}(\downarrow, \pi_1, \cap)$, which, moreover, are equivalent, by Corollary 5.20. In addition, they are also equivalent to $\mathcal{X}(\downarrow, \pi_1)$ and $\mathcal{C}(\downarrow, \pi_1)$, by Theorem 5.19. We refer to these languages as the *strictly downward positive (core) XPath algebra*. Combining the aforementioned results yields the following.

Theorem 5.33. *Consider the strictly downward positive (core) XPath algebra. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. Then $v_1 \equiv_{\text{exp}} v_2$ if and only if $v_1 \approx_\downarrow v_2$.*

We finally turn to characterizing navigational expressiveness. Proposition 5.9 and its proof, and hence also Corollary 5.10, carry over to the current setting.

Theorem 5.34. *Consider the strictly downward positive (core) XPath algebra. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, w_1, v_2, w_2 \in V$ be such that, for $i = 1, 2$, w_i is a descendant of v_i. Then, the property that, for each expression e, $(v_1, w_1) \in e(D)$ if and only if $(v_2, w_2) \in e(D)$, is equivalent to $(v_1, w_1) \cong_{\approx_\downarrow} (v_2, w_2)$.*

To derive a BP-result for the strictly downward positive (core) XPath algebra, we observe that Lemmas 5.11 and 5.12 carry over to the current context, provided we replace downward 1-equivalence by downward relatedness. Hence, Theorem 5.13 carries over to

Theorem 5.35. *Let $D = (V, Ed, r, \lambda)$ be a document, and let $R \subseteq V \times V$. Then, there exists an expression e in the strictly downward positive (core) XPath algebra such that $e(D) = R$ if and only if,*

1. *for all $v, w \in V$, $(v, w) \in R$ implies w is a descendant of v; and,*
2. *for all $v_1, w_1, v_2, w_2 \in V$ such that, for $i = 1.2.$ w_i a descendant of v_i and $(v_1, w_1) \cong_{\geq_\downarrow} (v_2, w_2)$, $(v_1, w_1) \in R$ implies $(v_2, w_2) \in R$.*

The major difference between Theorems 5.13 and 5.35 is that, in the former, R is a partition of maximal sets of $\equiv^k_\downarrow$-congruent nodes, while, in the latter, R is merely closed under $\geq_\downarrow$-congruence.

We can also recast Theorem 5.35 in terms of node-level navigation, in much the same way as Theorem 5.13.

Theorem 5.36. *Let $D = (V, Ed, r, \lambda)$ be a document, let $v \in V$, and let $W \subseteq V$. Then there exists an expression e in the strictly downward positive (core) XPath algebra such that $e(D)(v) = W$ if and only if all nodes of W are descendants of v, and, for all $w_1, w_2 \in V$ with $(v, w_1) \cong_{\geq_\downarrow} (v, w_2)$, $w_1 \in W$ implies $w_2 \in W$.*

Corollary 5.37. *Let $D = (V, Ed, r, \lambda)$ be a document, and let $W \subseteq V$. Then there exists an expression e in the strictly downward positive (core) XPath algebra such that $e(D)(r) = W$ if and only if, for all $w_1, w_2 \in V$ such that $w_1 \geq_\updownarrow w_2$, $w_1 \in W$ implies $w_2 \in W$.*

6. Weakly downward languages

We now turn to *weakly downward* languages: for any node v of a document D all nodes in $e(D)(v)$ are descendants of v, but there are possibly nodes v for which $e(D)(v) \neq e(D')(v)$, with D' the subtree of D rooted at v.

6.1. Sufficient conditions for expression-equivalence

The key notion in Sections 6.1–6.3 is $\equiv^k_\updownarrow$-congruence, $k \geq 1$, restricted to ancestor-descendant pairs. We first explore some properties of this notion, the first of which can be proved in a straightforward manner.

Lemma 6.1. *Let $D = (V, Ed, r, \lambda)$ be a document, let $v_1, w_1, v_2, w_2 \in V$ such that, for $i = 1, 2$, w_i is a descendant of v_i, and let $k \geq 1$. Then, $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$ if and only if $(v_1, w_1) \cong (v_2, w_2)$ and $w_1 \equiv^k_\updownarrow w_2$.*

Lemma 6.2. *Let $D = (V, Ed, r, \lambda)$ be a document, let $v_1, w_1 \in V$ be such that w_1 is a descendant of v_1, and let $k \geq 1$. Then, we have the following.*

1. *Let $v_2 \in V$ be such that $v_1 \equiv^k_\updownarrow v_2$. Then, v_2 has a descendant w_2 such that $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$.*
2. *Let $w_2 \in V$ be such that $w_1 \equiv^k_\updownarrow w_2$. Then, w_2 has an ancestor v_2 such that $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$.*

Proof. By Lemma 5.1 v_2 has a descendant w_2 such that $(v_1, w_1) \cong_{\equiv^k_\downarrow} (v_2, w_2)$. By Proposition 4.16, $(r, v_1) \cong_{\equiv^k_\downarrow} (r, v_2)$. It now readily follows that $(r, w_1) \cong_{\equiv^k_\downarrow} (r, w_2)$ and $w_1 \equiv^k_\updownarrow w_2$. Claim (1) now follows from Lemma 6.1.

Claim (2) can be shown by induction on the length of the path from v_1 to w_1. If $v_1 = w_1$, then obviously, $v_2 := w_2$. If $v_1 \neq w_1$, we have in particular that $w_1 \neq r$, and, hence, by $w_1 \equiv^k_\updownarrow w_2$, that $w_2 \neq r$. Let y_1 and y_2 be the parents of w_1 and w_2. By definition, $y_1 \equiv^k_\updownarrow y_2$, and, by the induction hypothesis, there exists $v_2 \in V$ such that $(v_1, y_1) \cong_{\equiv^k_\updownarrow} (v_2, y_2)$. It now readily follows that $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$. □

We now link $\equiv^k_\updownarrow$-congruence of ancestor-descendant pairs of nodes with expressibility in weakly downward languages.

Proposition 6.3. *Let $k \geq 1$, and let E be the set of all nonbasic operations in Table 1, except for upward navigation ("$\uparrow$"), inverse ("$.^{-1}$"), and selection on at least m children satisfying some condition ("$\text{ch}_{\geq m}(.)$") for $m > k$. Let e be an expression in $\mathcal{X}(E)$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, w_1, v_2, w_2 \in V$ be such that, for $i = 1, 2$, w_1 is a descendant of v_1, and $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$. Then, $(v_1, w_1) \in e(D)$ if and only if $(v_2, w_2) \in e(D)$.*

Proof. The proof goes along the same lines of the proof of Proposition 5.2, except, of course, for the inductive step for the second projection ("π_2"). Thus, let $e := \pi_2(f)$, with f satisfying Proposition 6.3. If $(v_1, w_1) \in \pi_2(f)$, then $v_1 = w_1$, and, hence, $v_2 = w_2$. Also, there exists $y_1 \in V$ such that $(y_1, v_1) \in f(D)$. By Lemma 6.2, (2), there exists $y_2 \in V$ such that $(y_1, v_1) \cong_{\equiv^k_\updownarrow} (y_2, v_2)$. By the induction hypothesis, $(y_2, v_2) \in f(D)$. Hence, $(v_2, v_2) \in \pi_2(f)(D)$. □

Combining Proposition 6.3 with Lemma 6.2, yields the following.

Corollary 6.4. *Let $k \geq 1$, and let E be the set of all nonbasic operations in Table 1, except for upward navigation ("$\uparrow$"), inverse ("$.^{-1}$"), and selection on at least m children ("$\mathrm{ch}_{\geq m}(.)$") for $m > k$. Let e be an expression in $\mathcal{X}(E)$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, w_1 \in V$ be such that w_1 is a descendant of v_1 and $(v_1, w_1) \in e(D)$. Then, we have the following.*

1. *Let $v_2 \in V$ be such that $v_1 \equiv^k_\updownarrow v_2$. Then, v_2 has a descendant w_2 such that $(v_2, w_2) \in e(D)$.*
2. *Let $w_2 \in V$ be such that $w_1 \equiv^k_\updownarrow w_2$. Then, w_2 has an ancestor v_2 such that $(v_2, w_2) \in e(D)$.*

Finally, we infer the following from Corollary 5.4, (1):

Corollary 6.5. *Let $k \geq 1$, and let E be a set of nonbasic operations not containing upward navigation ("$\uparrow$"), inverse ("$.^{-1}$"), and selection on at least m children satisfying some condition ("$\mathrm{ch}_{\geq m}(.)$") for $m > k$. Consider the language $\mathcal{X}(E)$ or $\mathcal{C}(E)$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$ If $v_1 \equiv^k_\updownarrow v_2$, then $v_1 \equiv_{\exp} v_2$.*

6.2. Necessary conditions for expression equivalence

We now explore requirements on the set of nonbasic operations expressible in a language under which downward-k-equivalence ($k \geq 1$) is necessary for expression-equivalence. As we tried to make as few assumptions as possible, Proposition 6.6 also holds for a class of languages that are *not* downward.

Proposition 6.6. *Let $k \geq 1$, and let E be a set of nonbasic operations containing at least one navigation operation ("$\downarrow$" or "$\uparrow$") and set difference ("$-$"). Consider the language $\mathcal{X}(E)$ or $\mathcal{C}(E)$. Assume that both projections ("π_1" and "π_2") are expressible, as well as selection on at least m children satisfying some condition ("$\mathrm{ch}_{\geq m}(.)$"), for $m = 1, \ldots, k$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. If $v_1 \equiv_{\exp} v_2$, then $v_1 \equiv^k_\updownarrow v_2$.*

Proof. Without loss of generality, we may assume that the language under consideration is $\mathcal{C}(E)$. In Proposition 5.6, we have already established that $v_1 \equiv_{\exp} v_2$ implies $v_1 \equiv^k_\downarrow v_2$. By induction on the length of the path from r to v_1, we establish that, furthermore, $v_1 \equiv^k_\updownarrow v_2$. For the basis of the induction, let $v_1 = r$. Let d be the height of D. We distinguish two cases:

1. $\downarrow \in E$. Then, $\downarrow^d(D)(v_1) \neq \emptyset$. Hence, $\downarrow^d(D)(v_2) \neq \emptyset$, and $v_2 = r$.
2. $\uparrow \in E$. Then, $\pi_2(\uparrow^d)(D)(v_1) \neq \emptyset$. Hence, $\pi_2(\uparrow^d)(D)(v_2) \neq \emptyset$, and $v_2 = r$.

So, $v_1 \equiv^k_\updownarrow v_2$. For the induction step, let $v_1 \neq r$. We again distinguish two cases:

1. $\downarrow \in E$. Then, $\pi_2(\downarrow)(D)(v_1) \neq \emptyset$, and, hence, $\pi_2(\downarrow)(D)(v_2) \neq \emptyset$. So, $v_2 \neq r$.
2. $\uparrow \in E$. Then, $\uparrow(D)(v_1) \neq \emptyset$, and, hence, $\uparrow(D)(v_2) \neq \emptyset$. So, $v_2 \neq r$.

Now, for $i = 1, 2$, let u_i be the parent of v_i. We show that $u_1 \equiv_{\exp} u_2$. Thereto, let e be an expression in $\mathcal{C}(E)$ for which $e(D)(u_1) \neq \emptyset$. Again, we distinguish two cases:

1. $\downarrow \in E$. Then, $\pi_2(\pi_1(e)/\downarrow)(v_1) \neq \emptyset$. Hence, $\pi_2(\pi_1(e)/\downarrow)(v_2) \neq \emptyset$, and $e(D)(u_2) \neq \emptyset$.
2. $\uparrow \in E$. Then, $\uparrow/e(v_1) \neq \emptyset$. Hence, $\uparrow/e(v_2) \neq \emptyset$, and $e(D)(u_2) \neq \emptyset$.

In both cases, the induction hypothesis yields $u_1 \equiv^k_\updownarrow u_2$. Hence, $v_1 \equiv^k_\updownarrow v_2$. □

We see that Proposition 6.6 is as well applicable to weakly downward languages as to weakly upward languages (see Section 7.2). We shall see in Section 7.2 that this is no coincidence. For now, we suffice with concluding that k-equivalence is necessary for expression-equivalence under a weakly downward language containing downward navigation ("$\downarrow$"), both projections ("π_1" and "π_2"), and set difference ("$-$"), provided selection on at least m children satisfying some condition ("$\mathrm{ch}_{\geq m}(.)$") for $m = 1, \ldots, k$ can be expressed.

6.3. Characterization of expression equivalence

We call $\mathcal{X}(\downarrow, \pi_1, \pi_2, \mathrm{ch}_{\geq 1}(.), \ldots, \mathrm{ch}_{\geq k}(.), -)$ ($\mathcal{C}(\downarrow, \pi_1, \pi_2, \mathrm{ch}_{\geq 1}(.), \ldots, \mathrm{ch}_{\geq k}(.), -)$) the *weakly downward* (*core*) *XPath algebra with counting up to* k. Moreover, they are equivalent, by Corollary 5.20. As these are the languages under consideration which effectively contain downward navigation ("$\downarrow$") and satisfy both Corollary 6.5 and Proposition 5.6, we obtain

Theorem 6.7. *Let $k \geq 1$, and consider the weakly downward (core) XPath algebra with counting up to k. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. Then $v_1 \equiv_{\exp} v_2$ if and only if $v_1 \equiv^k_\updownarrow v_2$.*

A special case arises when $k = 1$, since, by Proposition 2.4, selection on at least one child satisfying some condition ("$\text{ch}_{\geq 1}(.)$") can be expressed in terms of the other operations required by Theorem 6.7. We refer to the language $\mathcal{X}(\downarrow, \pi_1, \pi_2, -)$ ($\mathcal{C}(\downarrow, \pi_1, \pi_2, -)$) as the *weakly downward* (*core*) *XPath algebra*. We have the following.

Corollary 6.8. *Consider the weakly downward (core) XPath algebra. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. Then $v_1 \equiv_{\text{exp}} v_2$, if and only if $v_1 \equiv^1_\updownarrow v_2$.*

6.4. Characterization of navigational expressiveness

We start by proving a converse to Proposition 6.3.

Proposition 6.9. *Let $k \geq 1$, and let E be a set of nonbasic operations containing downward navigation ("$\downarrow$") and set difference ("$-$"). Consider the language $\mathcal{X}(E)$ or $\mathcal{C}(E)$. Assume that both projections ("π_1" and "π_2") are expressible, as well as selection on at least m children satisfying some condition ("$\text{ch}_{\geq m}(.)$"), for $m = 1, \ldots, k$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, w_1, v_2, w_2 \in V$ be such that, for $i = 1, 2$, w_i is a descendant of v_i, and, for each expression e, $(v_1, w_1) \in e(D)$ if and only if $(v_2, w_2) \in e(D)$. Then $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$.*

Proof. By Lemma 6.1, it suffices to prove that $w_1 \equiv^k_\updownarrow w_2$, or, by Proposition 4.16, that $(r, w_1) \cong_{\equiv^k_\downarrow} (r, w_2)$. By Proposition 5.9, $(v_1, w_1) \cong_{\equiv^k_\downarrow} (v_2, w_2)$. Hence, we still must show that $(r, v_1) \cong_{\equiv^k_\downarrow} (r, v_2)$. Since $(v_1, w_1) \in \pi_2(\text{sig}(r, v_1))/\text{sig}(v_1, w_1)$, $(v_2, w_2) \in \pi_2(\text{sig}(r, v_1))/\text{sig}(v_1, w_1)$, from which we readily deduce that $(r, v_1) \cong (r, v_2)$. Let u_1 be a node on the path from r to v_1, and let u_2 be the corresponding node on the path from r to v_2. Then, $(r, u_1) \cong (r, u_2)$ and $(u_1, v_1) \cong (u_2, v_2)$. Now, let f be any expression in the language such that $f(D)(u_1) \neq \emptyset$. Then, $(u_1, u_1) \in \pi_1(f)(D)$. Let $e := \pi_2(\pi_1(f)/\text{sig}(u_1, v_1))/\text{sig}(v_1, w_1)$. By construction, $(v_1, w_1) \in e(D)$. Hence, $(v_2, w_2) \in e(D)$, which implies $(u_2, u_2) \in \pi_1(f)(D)$ or $f(D)(u_2) \neq \emptyset$. The same holds vice versa, and we may thus conclude $u_1 \equiv_{\text{exp}} u_2$, and, hence, by Proposition 5.6, $u_1 \equiv^k_\downarrow u_2$. So, $(r, v_1) \equiv^k_\downarrow (r, v_2)$. □

Combining Propositions 6.3 and 6.9 yields the following.

Corollary 6.10. *Let $k \geq 1$, and consider the weakly downward (core) XPath algebra with counting up to k. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, w_1, v_2, w_2 \in V$ be such that, for $i = 1, 2$, w_i is a descendant of v_i. Then, the property that, for each expression e, $(v_1, w_1) \in e(D)$ if and only if $(v_2, w_2) \in e(D)$, is equivalent to the property $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$.*

From here on, the derivation of a BP-result for the weakly downward (core) XPath algebra with counting up to k follows the development in Section 5.4 very closely, which is why we only state the final results.

Theorem 6.11. *Let $k \geq 1$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $R \subseteq V \times V$. Then, there exists an expression e in the weakly downward (core) XPath algebra with counting up to k such that $e(D) = R$ if and only if,*

1. *for all $v, w \in V$, $(v, w) \in R$ implies w is a descendant of v; and,*
2. *for all $v_1, w_1, v_2, w_2 \in V$ such that, for $i = 1, 2$, w_i a descendant of v_i, and $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$, $(v_1, w_1) \in R$ implies $(v_2, w_2) \in R$.*

The specialization to the weakly downward (core) XPath algebra is

Corollary 6.12. *Let $D = (V, Ed, r, \lambda)$ be a document, and let $R \subseteq V \times V$. There exists an expression e in the weakly downward (core) XPath algebra such that $e(D) = R$ if and only if,*

1. *for all $v, w \in V$, $(v, w) \in R$ implies w is a descendant of v; and,*
2. *for all $v_1, w_1, v_2, w_2 \in V$ such that w_i a descendant of v_i and $(v_1, w_1) \cong_{\equiv^1_\updownarrow} (v_2, w_2)$, $(v_1, w_1) \in R$ implies $(v_2, w_2) \in R$.*

We recast Theorem 6.11 and Corollary 6.12 in terms of node-level navigation.

Theorem 6.13. *Let $k \geq 1$. Let $D = (V, Ed, r, \lambda)$ be a document, let $v \in V$, and let $W \subseteq V$. Then there exists an expression e in the weakly downward (core) XPath algebra with counting up to k such that $e(D)(v) = W$ if and only if all nodes of W are descendants of v, and, for all $w_1, w_2 \in W$ with $(v, w_1) \cong_{\equiv^k_\updownarrow} (v, w_2)$, $w_1 \in W$ implies $w_2 \in W$.*

Corollary 6.14. *Let $D = (V, Ed, r, \lambda)$ be a document, let $v \in V$, and let $W \subseteq V$. Then there exists an expression e in the weakly downward (core) XPath algebra such that $e(D)(v) = W$ if and only if all nodes of W are descendants of v, and, for all $w_1, w_2 \in W$ with $(v, w_1) \cong_{\equiv^1_\updownarrow} (v, w_2)$, $w_1 \in W$ implies $w_2 \in W$.*

For $v = r$, the condition $(v, w_1) \cong_{\equiv^k_\updownarrow} (v, w_2)$ reduces to $w_1 \equiv^k_\updownarrow w_2$, by Proposition 4.16 and Lemma 6.1. Comparing Theorem 6.13 and Corollary 6.14 with Theorem 5.17 and Corollary 5.18 then immediately yields the following.

Theorem 6.15. *Let $D = (V, Ed, r, \lambda)$.*

1. *for each expression e in the weakly downward (core) XPath algebra with counting up to k, $k \geq 1$, there exists an expression e' in the strictly downward (core) XPath algebra with counting up to k such that $e(D)(r) = e'(D)(r)$; in particular,*
2. *for each expression e in the weakly downward (core) XPath algebra, there exists an expression e' in the strictly downward (core) XPath algebra such that $e(D)(r) = e'(D)(r)$.*

Hence, the corresponding weakly downward and strictly downward languages are navigationally equivalent if navigation always starts from the root.

6.5. *Weakly downward languages not containing set difference*

To find characterizations for weakly downward languages not containing set difference, we can proceed in two ways:

1. we proceed as in Section 5.5.2 for strictly downward languages without set difference, i.e., reviewing the results in Sections 6.1–6.3 and examine to which extent these results in the case where $k = 1$ allow replacing 1-equivalence by relatedness (Definition 5.23); or
2. we start from the results in Section 5.5.2 on strictly downward languages without set difference and "bootstrap" them to results on weakly downward languages without set difference in the same way as the results on strictly downward languages with set difference in Sections 5.1–5.3 were bootstrapped to the results on weakly downward languages without set difference in Sections 6.1–6.3.

Of course, both approaches lead to the same results. As the necessary intermediate lemmas and all the proofs can readily be deduced in one of the two ways described above, we only give the main results.

Concretely, the languages for which we provide characterizations in this Section are $\mathcal{X}(\downarrow, \pi_1, \pi_2, \cap)$ and $\mathcal{C}(\downarrow, \pi_1, \pi_2, \cap)$, which, moreover, are equivalent, by Corollary 5.20. We refer to one of these languages as the *weakly downward positive (core) XPath algebra*. In addition, we can eliminate intersection. Hence, the weakly downward positive (core) XPath algebra is equivalent to $\mathcal{X}(\downarrow, \pi_1, \pi_2)$.

We now summarize the characterization results.

Theorem 6.16. *Consider the weakly downward positive (core) XPath algebra. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. Then $v_1 \equiv_{\exp} v_2$ if and only if $v_1 \approxeq_\updownarrow v_2$.*

Theorem 6.17. *Consider the weakly downward positive (core) XPath algebra. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, w_1, v_2, w_2 \in V$ be such that, for $i = 1, 2$, w_i is a descendant of v_i. Then, the property that, for each expression e, $(v_1, w_1) \in e(D)$ if and only if $(v_2, w_2) \in e(D)$, is equivalent to $(v_1, w_1) \cong_{\approxeq_\updownarrow} (v_2, w_2)$.*

Theorem 6.18. *Let $D = (V, Ed, r, \lambda)$ be a document, and let $R \subseteq V \times V$. Then, there exists an expression e in the weakly downward positive (core) XPath algebra such that $e(D) = R$ if and only if,*

1. *for all $v, w \in V$, $(v, w) \in R$ implies w is a descendant of v; and,*
2. *for all $v_1, w_1, v_2, w_2 \in V$ such that, for $i = 1, 2$, w_i a descendant of v_i, and $(v_1, w_1) \cong_{\geq_\updownarrow} (v_2, w_2)$, $(v_1, w_1) \in R$ implies $(v_2, w_2) \in R$.*

Corollary 6.19. *Let $D = (V, Ed, r, \lambda)$ be a document, let $v \in V$, and let $W \subseteq V$. Then there exists an expression e in the weakly downward positive (core) XPath algebra such that $e(D)(v) = W$ if and only if all nodes of W are descendants of v, and, for all $w_1, w_2 \in V$ such that $(v, w_1) \cong_{\geq_\updownarrow} (v, w_2)$, $w_1 \in W$ implies $w_2 \in W$.*

Corollary 6.20. *Let $D = (V, Ed, r, \lambda)$ be a document, and let $W \subseteq V$. Then there exists an expression e in the weakly downward positive (core) XPath algebra such that $e(D)(r) = W$ if and only if, for all $w_1, w_2 \in V$ such that $w_1 \geq_\updownarrow w_2$, $w_1 \in W$ implies $w_2 \in W$.*

Hence, the strictly and weakly downward positive (core) XPath algebra are navigationally equivalent if navigation always starts from the root.

7. Upward languages

We call a language *upward* if, for every expression e and for every node v of a document D, all nodes in $e(D)(v)$ are ancestors of v. If in addition it is always the case that $e(D)(v) = e(D')$, where D' is the subtree of D obtained by removing from D all strict descendants of v, we call the language *strictly upward*. Otherwise, we call it *weakly upward*. For E a set of nonbasic operations of Table 1, $\mathcal{X}(E)$ or $\mathcal{C}(E)$ is upward if it does not contain downward navigation ("$\downarrow$"), and inverse ("$.^{-1}$"). Additionally, $\mathcal{X}(E)$ or $\mathcal{C}(E)$ is strictly upward if it does not contain second projection ("π_2") and counting operations ("$\mathrm{ch}_{\geq k}(.)$").

Of course, there is a distinct asymmetry between strictly upward and downward languages: because a node has at most one parent, the analysis of strictly upward languages is much easier than that of strictly downward languages. We shall see, however, that this asymmetry disappears for their weak counterparts.

Finally, we observe that, for upward languages, set difference ("$-$") and intersection ("$\cap$") can still be eliminated, except where set difference operations are used as operands in boolean combinations of subexpressions inside a first projection. Hence, an upward language and its corresponding core language coincide.

7.1. Strictly upward languages

The languages we consider here are $\mathcal{X}(\uparrow, \pi_1, -)$ and $\mathcal{C}(\uparrow, \pi_1, -)$, which are equivalent, and $\mathcal{X}(\uparrow, \pi_1, \cap) = \mathcal{X}(\uparrow, \pi_1)$ and $\mathcal{C}(\uparrow, \pi_1, \cap) = \mathcal{C}(\uparrow, \pi_1)$, which are also equivalent. We refer to these as the *strictly upward (core) XPath algebra*, respectively the *strictly upward (core) positive XPath algebra*. We summarize the characterization results, as the proofs are similar to those in Section 5.

Theorem 7.1. *Consider the strictly upward (positive) (core) XPath algebra. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. Then $v_1 \equiv_{\exp} v_2$, if and only if $v_1 \equiv_{\uparrow} v_2$.*

Theorem 7.2. *Consider the strictly upward (positive) (core) XPath algebra. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, w_1, v_2, w_2 \in V$ be such that, for $i = 1, 2$, w_i is an ancestor of v_i. Then, the property that, for each expression e, $(v_1, w_1) \in e(D)$ if and only if $(v_2, w_2) \in e(D)$, is equivalent to $(v_1, w_1) \cong_{\equiv_{\uparrow}} (v_2, w_2)$.*

Theorem 7.3. *Let $D = (V, Ed, r, \lambda)$ be a document, and let $R \subseteq V \times V$. Then, there exists an expression e in the strictly upward (core) XPath algebra such that $e(D) = R$ if and only if,*

1. *for all $v, w \in V$, $(v, w) \in R$ implies w is an ancestor of v; and,*
2. *for all $v_1, w_1, v_2, w_2 \in V$ such that, for $i = 1, 2$, w_i is an ancestor of v_i, and $(v_1, w_1) \cong_{\equiv_{\uparrow}} (v_2, w_2)$, $(v_1, w_1) \in R$ implies $(v_2, w_2) \in R$.*

Theorem 7.4. *Let $D = (V, Ed, r, \lambda)$ be a document, and let $R \subseteq V \times V$. Then, there exists an expression e in the strictly upward positive (core) XPath algebra such that $e(D) = R$ if and only if,*

1. *for all $v, w \in V$, $(v, w) \in R$ implies w is an ancestor of v; and,*
2. *for all $v_1, w_1, v_2, w_2 \in V$ such that, for $i = 1, 2$, w_i is an ancestor of v_1, and $(v_1, w_1) \cong_{\geq_{\uparrow}} (v_2, w_2)$, $(v_1, w_1) \in R$ implies $(v_2, w_2) \in R$.*

Notice that the difference between the strictly downward (core) XPath algebra and the strictly downward positive (core) XPath algebra becomes only apparent in their BP-characterizations.

7.2. Weakly upward languages

Weakly upward languages are closely related to weakly downward languages:

Theorem 7.5. *Let E_{down} be a set of nonbasic operations in which first projection ("π_1") is expressible, but which does not contain upward navigation ("$\uparrow$") and inverse ("$.^{-1}$"). Let E_{up} be the set of nonbasic operations obtained from E by replacing downward ("$\downarrow$") by upward navigation, first by second projection, and second by first projection. Then, for each expression e in $\mathcal{X}(E_{\text{down}})$ ($\mathcal{C}(E_{\text{down}})$, $\mathcal{X}(E_{\text{up}})$, respectively $\mathcal{C}(E_{\text{up}})$), there is an expression e' in $\mathcal{X}(E_{\text{up}})$ ($\mathcal{C}(E_{\text{up}})$, $\mathcal{X}(E_{\text{down}})$, respectively $\mathcal{C}(E_{\text{down}})$) such that e^{-1} and e' are equivalent at the level of queries.*

Proof. If e is in $\mathcal{X}(E_{\text{down}})$ or $\mathcal{C}(E_{\text{down}})$, we first replace each subexpression $\text{ch}_{\geq k}(f)$ in which f is not a condition by $\text{ch}_{\geq k}(\pi_1(f))$. If e is in $\mathcal{X}(E_{\text{up}})$ or $\mathcal{C}(E_{\text{up}})$, we replace each subexpression $\text{ch}_{\geq k}(f)$ in which f is not a condition, i.e., f is of the form $c/\uparrow/g$ with c a condition and g an arbitrary subexpression, by $\pi_1(g)/\text{ch}_{\geq k}(c)$. We eliminate inverse ("$.^{-1}$") from e^{-1} as in Proposition 2.3, except that we replace subexpressions (1) $\text{ch}_{\geq k}(f)^{-1}$, with f a condition, by $\text{ch}_{\geq k}(f^{-1})$ rather than $\text{ch}_{\geq k}(f)$; (2) $\pi_1(e)^{-1}$ by $\pi_2(e^{-1})$ rather than $\pi_1(e)$; and (3) $\pi_2(e)^{-1}$ by $\pi_1(e^{-1})$ rather than $\pi_2(e)$. □

Theorem 7.5 has as immediate consequence that each characterization for a weakly downward language in Section 6—in each instance containing both projections—yields a characterization for the corresponding weakly upward language by replacing "descendant" by "ancestor". Similarly, each characterization for a *stricly* downward language in Section 5—in each instance containing only first projection—yields a characterization for the corresponding weakly upward language—in each instance containing only second projection.

Finally, applying Theorem 7.5 on the strictly upward languages considered in Section 7.1 yields characterizations for weakly downward languages containing only second projection, which were not considered in Section 6. In view of space considerations, we shall not discuss these languages, however.

8. Languages for two-way navigation

We finally consider languages which are neither downward nor upward, i.e., in which navigation in both directions ("↓" and "↑") is possible. A notable difference in this case is that standard languages no longer always coincide with their associated core languages in expressive power. Below we distinguish languages with and without difference. In the first case, we discuss the standard languages and the core languages separately (Sections 8.1 and 8.2). In the second case, there is no need for this distinction (Section 8.3).

8.1. Standard languages with difference for two-way navigation

First, we state analogues to Lemmas 6.1 and 6.2 for pairs of nodes that are *not* necessarily ancestor-descendant pairs.

Lemma 8.1. *Let $D = (V, Ed, r, \lambda)$ be a document, let $v_1, w_1, v_2, w_2 \in V$, and let $k \geq 1$. Then, $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$ if and only if $(v_1, w_1) \cong (v_2, w_2)$, $v_1 \equiv^k_\updownarrow v_2$, and $w_1 \equiv^k_\updownarrow w_2$.*

Proof. We only consider the "if." Since $(v_1, w_1) \cong (v_2, w_2)$, $(\text{top}(v_1, w_1), v_1) \cong (\text{top}(v_2, w_2), w_2)$. By Lemma 6.1, $(\text{top}(v_1, w_1), v_1) \cong_{\equiv^k_\updownarrow} (\text{top}(v_2, w_2), v_2)$. In the same way, we derive $(\text{top}(v_1, w_1), w_1) \cong_{\equiv^k_\updownarrow} (\text{top}(v_2, w_2), w_2)$. Proposition 3.5, (2), (3) and (4), then yields the desired result. □

Lemma 8.2. *Let $D = (V, Ed, r, \lambda)$ be a document, let $v_1, w_1 \in V$, and let $k \geq 2$.*

1. *For each $v_2 \in V$ such that $v_1 \equiv^k_\updownarrow v_2$ there exists $w_2 \in V$ such that $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$; and*
2. *For each $w_2 \in V$ for which $w_1 \equiv^k_\updownarrow w_2$ there exists $v_2 \in V$ such that $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$.*

Proof. We only prove (1); see Fig. 4. By Lemma 6.2, (2), there exists $t_2 \in V$ such that $(\text{top}(v_1, w_1), v_1) \cong_{\equiv^k_\updownarrow} (t_2, v_2)$. Hence, also $(v_1, \text{top}(v_1, w_1)) \cong_{\equiv^k_\updownarrow} (v_2, t_2)$. Let y_1 be the child of $\text{top}(v_1, w_1)$ on the path to w_1. Since $k \geq 2$, there is a child y_2 of t_2 such that (1) $y_1 \equiv^k_\updownarrow y_2$ and (2) y_2 is not on the path from t_2 to v_2.[9] By Lemma 6.2, (2), there exists $w_2 \in V$ such that $(y_1, w_1) \cong_{\equiv^k_\updownarrow} (y_2, w_2)$. In particular, $w_1 \equiv^k_\updownarrow w_2$. By construction, $t_2 = \text{top}(v_2, w_2)$, and, hence, $(v_1, w_1) \cong (v_2, w_2)$. The result now follows from Lemma 8.1. □

We need one more lemma, to deal with composition adequately:

Lemma 8.3. *Let $D = (V, Ed, r, \lambda)$ be a document, let $v_1, w_1, v_2, w_2 \in V$ be such that $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$, and let $k \geq 3$. Then, for each $y_1 \in V$, there exists $y_2 \in V$ such that $(v_1, y_1) \cong_{\equiv^k_\updownarrow} (v_2, y_2)$, and $(y_1, w_1) \cong_{\equiv^k_\updownarrow} (y_2, w_2)$.*

Proof. The proof is a case analysis. In each case description, we assume implicitly that cases already dealt with before are excluded.

[9] Mind that t_2 has two different k-equivalent children if $\text{top}(v_1, w_1)$ has.

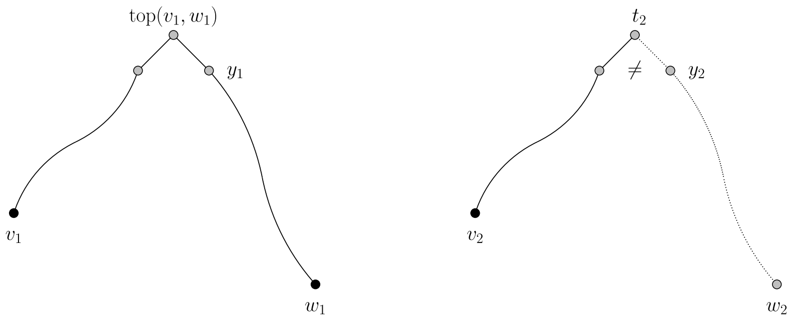


Fig. 4. Mutual position of the nodes in Lemma 8.2, (1), and its proof.

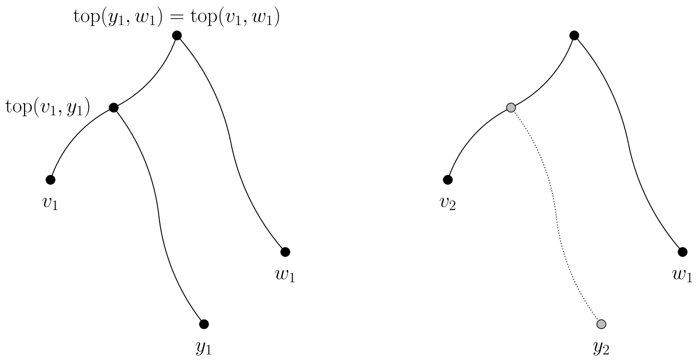


Fig. 5. Mutual position of the nodes in Case 5 of the proof of Lemma 8.3.

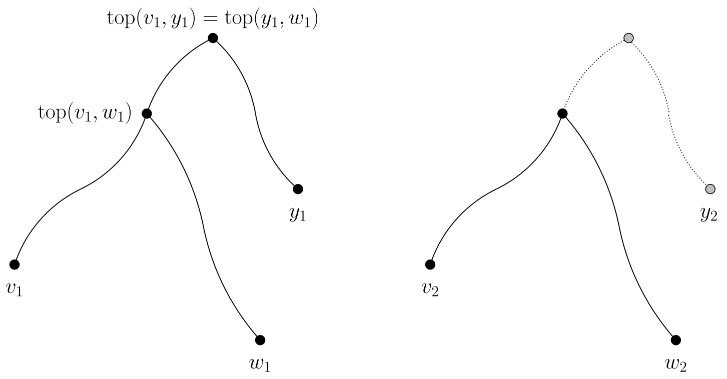
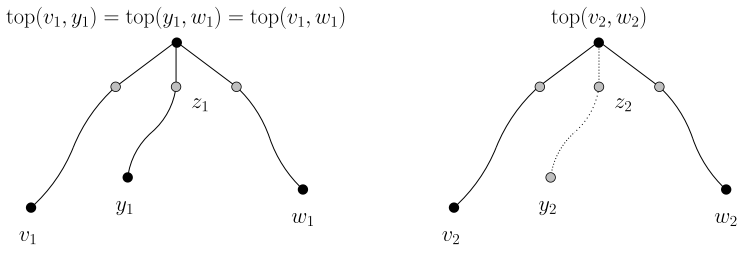


Fig. 6. Mutual position of the nodes in Case 7 of the proof of Lemma 8.3.

1. *y_1 is on the path from v_1 to w_1.* Clearly, y_2 is the node corresponding to y_1 on the path from v_2 to w_2.
2. *y_1 is a strict descendant of v_1.* By Lemma 6.2, (1), there is a (strict) descendant y_2 of v_2 such that $(v_1, y_1) \cong_{\equiv^k_\updownarrow} (v_2, y_2)$. The result now follows.
3. *y_1 is a strict descendant of w_1.* Analogous to the previous case.
4. *y_1 is a strict ancestor of* $\text{top}(v_1, w_1)$. By Lemma 6.2, (2), there is a (strict) ancestor y_2 of $\text{top}(v_2, w_2)$ such that $(\text{top}(v_1, w_1), y_1) \cong_{\equiv^k_\updownarrow} (\text{top}(v_2, w_2), y_2)$. The result now follows immediately.
5. $\text{top}(v_1, y_1)$ *is an internal node on the path from v_1 to* $\text{top}(v_1, w_1)$; see Fig. 5. By Lemma 8.2, (1), there exists $y_2 \in V$ such that $(v_1, y_1) \cong_{\equiv^k_\updownarrow} (v_2, y_2)$. Observe that $\text{top}(y_1, w_1) = \text{top}(v_1, w_1)$, an ancestor of v_1. Hence, by Proposition 3.5, (2)–(4), $(y_1, w_1) \cong (y_2, w_2)$. Since, moreover, $y_1 \equiv^k_\updownarrow y_2$ and $w_1 \equiv^k_\updownarrow w_2$, the desired result now follows from Lemma 8.1 and the constructions therein.
6. $\text{top}(y_1, w_1)$ *is an internal node on the path from* $\text{top}(v_1, w_1)$ *to w_1.* Analogous to the previous case.
7. $\text{top}(v_1, y_1) = \text{top}(y_1, w_1)$ *is a strict ancestor of* $\text{top}(v_1, w_1)$; see Fig. 6. By Lemma 8.2, (1), there exists $y_2 \in V$ such that $(v_1, y_1) \cong_{\equiv^k_\updownarrow} (v_2, y_2)$. Observe that $\text{top}(y_1, w_1)$ is an ancestor of v_1. Hence, by Proposition 3.5, (2)–(4), $(y_1, w_1) \cong (y_2, w_2)$. Since, moreover, $y_1 \equiv^k_\updownarrow y_2$ and $w_1 \equiv^k_\updownarrow w_2$, the desired result now follows from Lemma 8.1.



$\mathrm{top}(v_1, y_1) = \mathrm{top}(y_1, w_1) = \mathrm{top}(v_1, w_1)$
z_1
v_1
y_1
w_1
$\mathrm{top}(v_2, w_2)$
z_2
v_2
y_2
w_2

Fig. 7. Mutual position of the nodes in Case 8 of the proof of Lemma 8.3.

8. $\mathrm{top}(v_1, y_1) = \mathrm{top}(y_1, w_1) = \mathrm{top}(v_1, w_1)$; see Fig. 7. Let z_1 be the child of $\mathrm{top}(v_1, w_1)$ on the path to y_1. By assumption, z_1 is not on the path from v_1 to w_1. Since $k \geq 3$, there exists $z_2 \in V$ not on the path from v_2 to w_2 such that $z_1 \equiv^k_\updownarrow z_2$.[10] By Lemma 8.2, (1), there exists $y_2 \in V$ such that $(z_1, y_1) \cong_{\equiv^k_\updownarrow} (z_2, y_2)$. The result now follows readily. □

We are now ready to state the analogue of Proposition 6.3 for languages with two-way navigation. The proof is similar.

Proposition 8.4. *Let $k \geq 3$, and let E be the set of all nonbasic operations in Table 1, except for selection on at least m children satisfying some condition ("$\mathrm{ch}_{\geq m}(.)$") for $m > k$. Let e be an expression in $\mathcal{X}(E)$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, w_1, v_2, w_2 \in V$ be such that $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$. Then, $(v_1, w_1) \in e(D)$ if and only if $(v_2, w_2) \in e(D)$.*

As in Section 6.2, we can infer the following result from Proposition 8.4.

Corollary 8.5. *Let $k \geq 3$, and let E be a set of nonbasic operations not containing selection on at least m children satisfying some condition ("$\mathrm{ch}_{\geq m}(.)$") for $m > k$. Consider the language $\mathcal{X}(E)$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. If $v_1 \equiv^k_\updownarrow v_2$, then $v_1 \equiv_{\exp} v_2$.*

The standard language for two-way navigation satisfying both Corollary 8.5 and Proposition 6.6—which is indeed also applicable to an important class of languages allowing two-way navigation—is $\mathcal{X}(\downarrow, \uparrow, \mathrm{ch}_{\geq 1}(.), \ldots, \mathrm{ch}_{\geq k}(.), -)$.[11] We call this language the *XPath algebra with counting up to k*. Combining the aforementioned results yields the following.

Theorem 8.6. *Let $k \geq 3$, and consider the XPath algebra with counting up to k. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. Then, $v_1 \equiv_{\exp} v_2$ if and only if $v_1 \equiv^k_\updownarrow v_2$.*

By Proposition 2.4, selection on up to three children satisfying some condition ("$\mathrm{ch}_{\geq m}(.)$," $1 \leq m \leq 3$) can be expressed in the XPath algebra. Hence, a special case arises for $k = 3$:

Corollary 8.7. *Consider the XPath algebra. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. Then, $v_1 \equiv_{\exp} v_2$ if and only if $v_1 \equiv^3_\updownarrow v_2$.*

We next prove a converse to Proposition 8.4.

Proposition 8.8. *Let $k \geq 3$, and consider the XPath algebra with counting up to k. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, w_1, v_2, w_2 \in V$ be such that, for each expression e, $(v_1, w_1) \in e(D)$ if and only if $(v_2, w_2) \in e(D)$. Then $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$.*

Proof. Using signature expressions, we can see that $(v_1, w_1) \cong (v_2, w_2)$. Now, let f be any expression such that $f(D)(v_1) \neq \emptyset$, and let $e := \pi_1(f) / \mathrm{sig}(v_1, w_1)$. By construction, $(v_1, w_1) \in e(D)$. Hence, $(v_2, w_2) \in e(D)$, and $f(D)(v_2) \neq \emptyset$. As the same holds vice versa, $v_1 \equiv_{\exp} v_2$. Hence, by Theorem 8.6, $v_1 \equiv^k_\updownarrow v_2$. A similar argument yields $w_1 \equiv^k_\updownarrow w_2$. By Lemma 8.1, $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$. □

Combining Propositions 8.4 and 8.8 yields the following characterization.

Corollary 8.9. *Let $k \geq 3$, and consider the XPath algebra with counting up to k. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, w_1, v_2, w_2 \in V$. Then, the property that, for each expression e, $(v_1, w_1) \in e(D)$ if and only if $(v_2, w_2) \in e(D)$, is equivalent to $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$.*

[10] Mind that $\mathrm{top}(v_2, w_2)$ has two or three different k-equivalent children if $\mathrm{top}(v_1, w_1)$ has.

[11] All other operations are redundant, by Proposition 2.3.

Using Theorem 8.6 instead of Theorem 5.7, we can recast the proof of Lemma 5.11 into a proof of

Lemma 8.10. *Let $k \geq 3$. Let $D = (V, Ed, r, \lambda)$ be a document, and let v_1 be a node of D. There exists an expression e_{v_1} in the XPath algebra with counting up to k such that, for each node v_2 of D, $e_{v_1}(D)(v_2) \neq \emptyset$ if and only if $v_1 \equiv^k_\updownarrow v_2$.*

We can next bootstrap Lemma 8.10 to the following result.

Lemma 8.11. *Let $k \geq 3$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, w_1 \in V$. There exists an expression e_{v_1,w_1} in the XPath algebra with counting up to k such that, for all $v_2, w_2 \in V$, $(v_2, w_2) \in e_{v_1,w_1}(D)$ if and only if $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$.*

Proof. From Lemma 8.10, we know that, for all $y_1 \in V$, there exists an expression e_{y_1} in the XPath algebra with counting up to k such that, for all $y_2 \in V$, $e_{y_1}(D)(y_2) \neq \emptyset$ if and only if $y_1 \equiv^k_\updownarrow y_2$. Now, denote $\mathrm{sig}(v_1, w_1) := \uparrow^u/\downarrow^d$, with $u, d \geq 0$, and define $e_{v_1,w_1} := \pi_1(e_{v_1})/\mathrm{sig}(v_1, w_1)/\pi_1(e_{w_1}) - \uparrow^{u-1}/\downarrow^{d-1}$ (where $\uparrow^{-1} = \downarrow^{-1} := \emptyset$). Let $v_2, w_2 \in V$ be such that $(v_2, w_2) \in e_{v_1,w_1}(D)$. Then, by Proposition 3.4, $\mathrm{sig}(v_1, w_1) = \mathrm{sig}(v_2, w_2)$. Furthermore, $(v_2, v_2) \in e_{v_1}(D)$ and $(w_2, w_2) \in e_{w_1}(D)$. By Lemma 8.10, $v_1 \equiv^k_\updownarrow v_2$ and $w_1 \equiv^k_\updownarrow w_2$. By Lemma 8.1, $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$. Clearly, Corollary 8.9 yields the converse. □

The BP characterization results now follow readily.

Theorem 8.12. *Let $k \geq 3$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $R \subseteq V \times V$. Then, there exists an expression e in the XPath algebra with counting up to k such that $e(D) = R$ if and only if, for all $v_1, w_1, v_2, w_2 \in V$ with $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$, $(v_1, w_1) \in R$ implies $(v_2, w_2) \in R$.*

The specialization to the XPath algebra is as follows.

Corollary 8.13. *Let $D = (V, Ed, r, \lambda)$ be a document, and let $R \subseteq V \times V$. There exists an expression e in the XPath algebra such that $e(D) = R$ if and only if, for all $v_1, w_1, v_2, w_2 \in V$ with $(v_1, w_1) \cong_{\equiv^3_\updownarrow} (v_2, w_2)$, $(v_1, w_1) \in R$ implies $(v_2, w_2) \in R$.*

We recast Theorem 8.12 and Corollary 8.13 in terms of node-level navigation.

Theorem 8.14. *Let $k \geq 3$. Let $D = (V, Ed, r, \lambda)$ be a document, let $v \in V$, and let $W \subseteq V$. Then there exists an expression e in the XPath algebra with counting up to k such that $e(D)(v) = W$ if and only if, for all $w_1, w_2 \in W$ with $(v, w_1) \cong_{\equiv^k_\updownarrow} (v, w_2)$, $w_1 \in W$ implies $w_2 \in W$.*

The specialization to the XPath algebra is as follows.

Corollary 8.15. *Let $D = (V, Ed, r, \lambda)$ be a document, let $v \in V$, and let $W \subseteq V$. Then there exists an expression e in the XPath algebra such that $e(D)(v) = W$ if and only if, for all $w_1, w_2 \in W$ with $(v, w_1) \cong_{\equiv^3_\updownarrow} (v, w_2)$, $w_1 \in W$ implies $w_2 \in W$.*

Finally, we consider the special case where navigation starts from the root. For $v = r$, the condition $(v, w_1) \cong_{\equiv^k_\updownarrow} (v, w_2)$ reduces to $w_1 \equiv^k_\updownarrow w_2$, by Proposition 4.16 and Lemma 6.1. Comparing Theorem 8.14 and Corollary 8.15 with, respectively, Theorem 5.17 and Corollary 5.18 then immediately yields

Theorem 8.16. *Let $D = (V, Ed, r, \lambda)$.*

1. *For each expression e in the XPath algebra with counting up to k, $k \geq 3$, there exists an expression e' in the strictly downward (core) XPath algebra with counting up to k such that $e(D)(r) = e'(D)(r)$.*
2. *For each expression e in the XPath algebra, there exists an expression e' in the strictly downward (core) XPath algebra with counting up to 3 such that $e(D)(r) = e'(D)(r)$.*

For navigation from the root, the only capability that the full XPath algebra adds compared to the strictly downward (core) XPath algebra (Theorem 6.15) is selection on at least 2 and at least 3 children satisfying some condition.

8.2. Core languages with difference for two-way navigation

We now investigate what changes if we replace a standard language with difference for two-way navigation by the corresponding core language. The most important observation is that both languages are no longer equivalent.

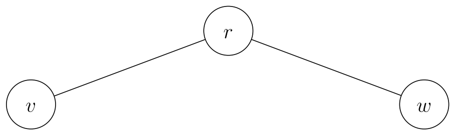


Fig. 8. Document of Example 8.17.

Example 8.17. Let $D = (V, Ed, r, \lambda)$ be the document in Fig. 8, and consider the expression $e := \uparrow/\downarrow - \varepsilon$ in the XPath algebra with counting up to k, $k \geq 2$. We have that $e(D) = \{(v, w), (w, v)\}$. From Proposition 8.19, it will follow, however, that, for every expression e' in the corresponding core XPath algebra with counting up to k, $(v, w) \in e'(D)$ implies that not only $(w, v) \in e'(D)$, but also $(v, v) \in e'(D)$ and $(w, w) \in e'(D)$.

We now explore which changes occur compared to Section 8.1. As Example 8.17 suggests, there is no hope that we can express congruence in the core XPath algebra with counting up to k, for any $k \geq 2$. Therefore, we shall have to work with subsumption instead of congruence. Luckily, Lemmas 8.1–8.3 still holds if we replace congruence by subsumption, except that we can strengthen the statement of the last one, as follows.

Lemma 8.18. *Let $D = (V, Ed, r, \lambda)$ be a document, let $v_1, w_1, v_2, w_2 \in V$ be such that $(v_1, w_1) \gtrsim_{\equiv^k_\updownarrow} (v_2, w_2)$, and let $k \geq 2$. Then, for each $y_1 \in V$, there exists $y_2 \in V$ such that $(v_1, y_1) \gtrsim_{\equiv^k_\updownarrow} (v_2, y_2)$, and $(y_1, w_1) \gtrsim_{\equiv^k_\updownarrow} (y_2, w_2)$.*

Proof. The only case in the proof of Lemma 8.3 where we used $k \geq 3$ is Case 8 $(\text{top}(v_1, y_1) = \text{top}(y_1, w_1) = \text{top}(v_1, w_1))$ to guarantee that the paths from $\text{top}(v_2, w_2)$ to v_2, w_2, and y_2 have no overlap. As this is no concern anymore when we consider subsumption rather than congruence, the condition $k \geq 2$ suffices to recast the proof of Lemma 8.3 into a proof of Lemma 8.18. $\square$

Lemma 8.3 was used to complete the induction step for composition ("/") in the proof of Proposition 8.4. If we replace Lemma 8.3 by Lemma 8.18, we can also avoid making the assumption $k \geq 3$ here. The restricted use of difference in core languages allows us to get away with subsumption instead of congruence.

Proposition 8.19. *Let $k \geq 2$, and let E be the set of all nonbasic operations in Table 1, except for selection on at least m children satisfying some condition ("$\text{ch}_{\geq m}(.)$") for $m > k$. Let e be an expression in $\mathcal{C}(E)$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, w_1, v_2, w_2 \in V$ be such that $(v_1, w_1) \gtrsim_{\equiv^k_\updownarrow} (v_2, w_2)$. Then, $(v_1, w_1) \in e(D)$ implies $(v_2, w_2) \in e(D)$.*

Proof. The proof is similar to that of Proposition 8.4, except that, in the induction step, we need not consider the case of set difference ("$-$"). However, we must consider instead the case of projection, where, in the expression $e := \pi_1(f)$ or $e := \pi_2(f)$, f is now a boolean combination of expressions. For reasons of symmetry, we only consider first projection. Since, without loss of generality, we may assume that f is union-free, we can write $f = f_1 \cap \ldots f_p \cap \overline{g_1} \cap \ldots \overline{g_q}$ for some $p \geq 1$ and $q \geq 0$, with $f_1, \ldots, f_p, g_1, \ldots, g_q$ in $\mathcal{C}(E)$ and satisfying the induction hypothesis. Here, $\overline{g}$, the complement of g, is defined by $\overline{g}(D) := V \times V - g(D)$. In particular, if $(v_1, w_1) = (v_1, v_1) \in \pi_1(f)(D)$, there exists $y_1 \in V$ such that, for $i = 1, \ldots, p$, $(v_1, y_1) \in f_i(D)$, and, for $j = 1, \ldots, q$, $(v_1, y_1) \notin g_j(D)$. By Lemma 8.2, there exists $y_2 \in V$ such that $(v_1, y_1) \cong_{\equiv^k_\updownarrow} (v_2, y_2)$. Hence, for $i = 1, \ldots, p$, $(v_2, y_2) \in f_i(D)$, by the induction hypothesis. Now, assume that, for some j, $1 \leq j \leq q$, $(v_2, y_2) \in g_j(D)$. Then, again by the induction hypothesis, $(v_1, y_1) \in g_j(D)$, a contradiction. Hence, for $j = 1, \ldots, q$, $(v_2, y_2) \notin g_j(D)$, and $(v_2, v_2) = (v_2, w_2) \in \pi_1(f)$. $\square$

By applying Proposition 8.19 twice, we obtain the following.

Corollary 8.20. *Let $k \geq 2$, and let E be the set of all nonbasic operations in Table 1, except for selection on at least m children satisfying some condition ("$\text{ch}_{\geq m}(.)$") for $m > k$. Let e be an expression in $\mathcal{C}(E)$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, w_1, v_2, w_2 \in V$ be such that $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$. Then, $(v_1, w_1) \in e(D)$ if and only if $(v_2, w_2) \in e(D)$.*

As in Section 6.2, we can infer the following result from Corollary 8.20.

Corollary 8.21. *Let $k \geq 2$, and let E be a set of nonbasic operations not containing selection on at least m children satisfying some condition ("$\text{ch}_{\geq m}(.)$") for $m > k$. Consider the language $\mathcal{C}(E)$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. If $v_1 \equiv^k_\updownarrow v_2$, then $v_1 \equiv_{\text{exp}} v_2$.*

Notice that, for $k \geq 3$, Corollary 8.21 also follows from Corollary 8.5. The standard language for two-way navigation satisfying both Corollary 8.5 and Proposition 6.6—which is indeed also applicable to an important class of languages allowing two-way navigation—is $\mathcal{C}(\downarrow, \uparrow, \pi_1, \pi_2, \mathrm{ch}_{\geq 1}(.), \dots, \mathrm{ch}_{\geq k}(.), -)$.[12] We call this language the *core XPath algebra with counting up to k*. Combining the aforementioned results yields the following.

Theorem 8.22. *Let $k \geq 2$, and consider the core XPath algebra with counting up to k. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. Then, $v_1 \equiv_{\exp} v_2$ if and only if $v_1 \equiv^k_\updownarrow v_2$.*

By Proposition 2.4, selection on up to two children satisfying some condition ("$\mathrm{ch}_{\geq m}(.)$," $1 \leq m \leq 2$) can be expressed in the core XPath algebra. Hence, a special case arises for $k = 2$:

Corollary 8.23. *Consider the core XPath algebra. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. Then, $v_1 \equiv_{\exp} v_2$ if and only if $v_1 \equiv^2_\updownarrow v_2$.*

The converse of Proposition 8.19 can be proved similarly as Proposition 8.8.

Proposition 8.24. *Let $k \geq 2$, and consider the core XPath algebra with counting up to k. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, w_1, v_2, w_2 \in V$ be such that, for each expression e, $(v_1, w_1) \in e(D)$ implies $(v_2, w_2) \in e(D)$. Then $(v_1, w_1) \gtrsim_{\equiv^k_\updownarrow} (v_2, w_2)$.*

Combining Propositions 8.19 and 8.24 yields the following characterization.

Corollary 8.25. *Let $k \geq 2$, and consider the core XPath algebra with counting up to k. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, w_1, v_2, w_2 \in V$. Then,*

1. *the property that, for each expression e, $(v_1, w_1) \in e(D)$ implies $(v_2, w_2) \in e(D)$, is equivalent to $(v_1, w_1) \gtrsim_{\equiv^k_\updownarrow} (v_2, w_2)$; and,*
2. *the property that, for each expression e, $(v_1, w_1) \in e(D)$ if and only if $(v_2, w_2) \in e(D)$, is equivalent to $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$.*

Lemma 8.10 also holds for the core XPath algebra, with the condition $k \geq 3$ replaced by $k \geq 2$. Lemma 8.11 is a different story, unfortunately. Example 8.17 already indicates that, given $v_1, w_1, v_2, w_2 \in V$, we can in general not hope for an expression e_{v_1,w_1} such that $(v_2, w_2) \in e_{v_1,w_1}(D)$ if and only if $(v_1, w_1) \cong_{\equiv^k_\updownarrow} (v_2, w_2)$. However, we can replace congruence by subsumption:

Lemma 8.26. *Let $k \geq 2$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, w_1 \in V$. There exists an expression e_{v_1,w_1} in the core XPath algebra with counting up to k such that, for all $v_2, w_2 \in V$, $(v_2, w_2) \in e_{v_1,w_1}(D)$ if and only if $(v_1, w_1) \gtrsim_{\equiv^k_\updownarrow} (v_2, w_2)$.*

Proof. The proof is very similar to that of Proposition 8.11, except that, from the proposed expression, the minus term must be omitted. □

The BP characterization results now follow readily.

Theorem 8.27. *Let $k \geq 2$. Let $D = (V, Ed, r, \lambda)$ be a document, and let $R \subseteq V \times V$. Then, there exists an expression e in the core XPath algebra with counting up to k such that $e(D) = R$ if and only if, for all $v_1, w_1, v_2, w_2 \in V$ such that $(v_1, w_1) \gtrsim_{\equiv^k_\updownarrow} (v_2, w_2)$, $(v_1, w_1) \in R$ implies $(v_2, w_2) \in R$.*

The specialization to the core XPath algebra is as follows.

Corollary 8.28. *Let $D = (V, Ed, r, \lambda)$ be a document, and let $R \subseteq V \times V$. There exists an expression e in the core XPath algebra such that $e(D) = R$ if and only if, for all $v_1, w_1, v_2, w_2 \in V$ such that $(v_1, w_1) \gtrsim_{\equiv^3_\updownarrow} (v_2, w_2)$, $(v_1, w_1) \in R$ implies $(v_2, w_2) \in R$.*

We recast Theorem 8.27 and Corollary 8.28 in terms of node-level navigation.

Theorem 8.29. *Let $k \geq 2$. Let $D = (V, Ed, r, \lambda)$ be a document, let $v \in V$, and let $W \subseteq V$. Then there exists an expression e in the core XPath algebra with counting up to k such that $e(D)(v) = W$ if and only if, for all $w_1, w_2 \in W$ such that $(v, w_1) \gtrsim_{\equiv^k_\updownarrow} (v, w_2)$, $w_1 \in W$ implies $w_2 \in W$.*

The specialization to the core XPath algebra is as follows.

[12] Inverse ("$^{-1}(.)$") is redundant, as it can be eliminated (cf. Proposition 2.3).

Corollary 8.30. *Let $D = (V, Ed, r, \lambda)$ be a document, let $v \in V$, and let $W \subseteq V$. Then there exists an expression e in the core XPath algebra such that $e(D)(v) = W$ if and only if, for all $w_1, w_2 \in W$ with $(v, w_1) \gtrsim_{\equiv^2_\updownarrow} (v, w_2)$, $w_1 \in W$ implies $w_2 \in W$.*

Finally, for the special case where navigation starts from the root, Theorem 8.29 and Corollary 8.30 reduce to the following.

Theorem 8.31. *Let $D = (V, Ed, r, \lambda)$.*

1. *For each expression e in the core XPath algebra with counting up to k, $k \geq 2$, there exists an expression e' in the strictly downward (core) XPath algebra with counting up to k such that $e(D)(r) = e'(D)(r)$.*
2. *For each expression e in the core XPath algebra, there exists an expression e' in the strictly downward (core) XPath algebra with counting up to 2 such that $e(D)(r) = e'(D)(r)$.*

For navigation from the root, the only capability that the full XPath algebra adds compared to the strictly downward (core) XPath algebra (Theorem 6.15) is selection on at least 2 children satisfying some condition.

8.3. Languages without difference for two-way navigation

As before with languages not containing difference, we do not consider counting operations. Taking into account Proposition 2.3, and recognizing that the techniques used in this paper to establish characterizations heavily use intersection, we only need this means that we only need to consider $\mathcal{X}(\downarrow, \uparrow, \cap) = \mathcal{X}(\downarrow, \uparrow, \pi_1, \pi_2, \cap)$ and $\mathcal{C}(\downarrow, \uparrow, \pi_1, \pi_2, \cap)$. We refer to these as the *positive* (*core*) *XPath algebra*. Some of the present authors showed [34] that $\mathcal{X}(\downarrow, \uparrow, \cap)$ and $\mathcal{X}(\downarrow, \uparrow, \pi_1, \pi_2)$ are equivalent in expressive power at the level of queries. Since obviously $\mathcal{X}(\downarrow, \uparrow, \pi_1, \pi_2) = \mathcal{C}(\downarrow, \uparrow, \pi_1, \pi_2)$, it follows readily that the positive XPath algebra and the positive core XPath algebra are equivalent. The following results, shown in [34], are only repeated for completeness' sake.

Theorem 8.32. *Consider the positive (core) XPath algebra. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, v_2 \in V$. Then, $v_1 \geq_{\text{exp}} v_2$ if and only if $v_1 \geq^1_\updownarrow v_2$, and $v_1 \equiv_{\text{exp}} v_2$ if and only if $v_1 \approxeq_\updownarrow v_2$.*

Theorem 8.33. *Consider the positive (core) XPath algebra. Let $D = (V, Ed, r, \lambda)$ be a document, and let $v_1, w_1, v_2, w_2 \in V$. Then,*

1. *the property that, for each expression e, $(v_1, w_1) \in e(D)$ implies $(v_2, w_2) \in e(D)$, is equivalent to $(v_1, w_1) \gtrsim_{\geq^1_\updownarrow} (v_2, w_2)$; and*
2. *the property that, for each expression e, $(v_1, w_1) \in e(D)$ if and only if $(v_2, w_2) \in e(D)$, is equivalent to $(v_1, w_1) \cong_{\approxeq_\updownarrow} (v_2, w_2)$.*

As in Section 6.5, we can bootstrap these results to BP-characterizations.

Theorem 8.34. *Let $D = (V, Ed, r, \lambda)$ be a document, and let $R \subseteq V \times V$. Then, there exists an expression e in the positive (core) XPath algebra such that $e(D) = R$ if and only if, for all $v_1, w_1, v_2, w_2 \in V$ with $(v_1, w_1) \gtrsim_{\geq_\updownarrow} (v_2, w_2)$, $(v_1, w_1) \in R$ implies $(v_2, w_2) \in R$.*

Finally, Theorem 8.34 can be specialized to the node level, as follows.

Corollary 8.35. *Let $D = (V, Ed, r, \lambda)$ be a document, let $v \in V$, and let $W \subseteq V$. Then there exists an expression e in the positive (core) XPath algebra such that $e(D)(v) = W$ if and only if, for all nodes w_1 and w_2 of D with $(v, w_1) \gtrsim_{\geq_\updownarrow} (v, w_2)$, $w_1 \in W$ implies $w_2 \in W$.*

Corollary 8.36. *Let $D = (V, Ed, r, \lambda)$ be a document, and let $W \subseteq V$. Then there exists an expression e in the positive (core) XPath algebra such that $e(D)(r) = W$ if and only if, for all $w_1, w_2 \in V$ with $w_1 \geq_\updownarrow w_2$, $w_1 \in W$ implies $w_2 \in W$.*

Hence, the positive (core) XPath algebra, the weakly downward positive (core) XPath algebra, and the strictly downward positive (core) XPath algebra are all navigationally equivalent if navigation always starts from the root.

9. Discussion

In this paper, we characterized the expressive power of several natural fragments of XPath at the document level, as summarized in Table 5. Of course, it is possible to consider other fragments or extensions of the XPath algebra and its data model. Analyzing these using our two-step methodology in order to further improve our understanding of the instance expressivity of Tarski's algebra is one possible research direction which we have pursued recently [12–15,34].

Table 5
Summary of main results.

Language	Node Relationship	Node Coupling Result	Path Relationship	Path Coupling Result	BP Result
Strictly downward (core) XPath algebra with counting up to k	$\equiv^k_\downarrow$	Theorem 5.7	$\cong_{\equiv^k_\downarrow}$	Corollary 5.10	Theorem 5.13
Strictly downward (core) XPath algebra	$\equiv^1_\downarrow$	Corollary 5.8	$\cong_{\equiv^1_\downarrow}$	Corollary 5.10	Corollary 5.14
Strictly downward positive (core) XPath algebra	$\approxeq_\downarrow$	Theorem 5.33	$\cong_{\approxeq_\downarrow}$	Theorem 5.34	Theorem 5.35
Weakly downward (core) XPath algebra with counting up to k	$\equiv^k_\updownarrow$	Theorem 6.7	$\cong_{\equiv^k_\updownarrow}$	Corollary 6.10	Theorem 6.11
Weakly downward (core) XPath algebra	$\equiv^1_\updownarrow$	Corollary 6.8	$\cong_{\equiv^1_\updownarrow}$	Corollary 6.10	Corollary 6.12
Weakly downward positive (core) XPath algebra	$\approxeq_\updownarrow$	Theorem 6.16	$\cong_{\approxeq_\updownarrow}$	Theorem 6.17	Theorem 6.18
Strictly upward (core) XPath algebra	$\equiv_\uparrow$	Theorem 7.1	$\cong_{\equiv_\uparrow}$	Theorem 7.2	Theorem 7.3
Strictly upward positive (core) XPath algebra	$\equiv_\uparrow$	Theorem 7.1	$\cong_{\equiv_\uparrow}$	Theorem 7.2	Theorem 7.4
Weakly upward languages	See Section 7.2				
XPath algebra with counting up to k	$\equiv^k_\updownarrow$	Theorem 8.6	$\cong_{\equiv^k_\updownarrow}$	Corollary 8.9	Theorem 8.12
XPath algebra	$\equiv^3_\updownarrow$	Corollary 8.7	$\cong_{\equiv^3_\updownarrow}$	Corollary 8.9	Corollary 8.13
Core XPath algebra with counting up to k	$\equiv^k_\updownarrow$	Theorem 8.22	$\cong_{\equiv^k_\updownarrow}$	Corollary 8.25	Theorem 8.27
Core XPath algebra	$\equiv^2_\updownarrow$	Corollary 8.23	$\cong_{\equiv^2_\updownarrow}$	Corollary 8.25	Corollary 8.28
Positive (core) XPath algebra [34]	$\approxeq_\updownarrow$	Theorem 8.32	$\cong_{\approxeq_\updownarrow}$	Theorem 8.33	Theorem 8.34

Another future research direction is refining the links between XPath and finite-variable first-order logics [35]. Indeed, such links have been established at the level of query semantics. For example, Marx [36] has shown that an extended version of Core XPath is equivalent to $\mathrm{FO}^2_{\mathtt{tree}}$—first-order logic using at most two variables over *ordered* node-labeled trees—interpreted in the signature `child`, `descendant`, and `following_sibling`.

Our results establish new links to finite-variable first-order logics at the document level. For example, we can show that, on a given document, the XPath algebra and FO^3—first-order logic with at most three variables—are equivalent in expressive power. Indeed, as we discussed above, at the document level, the XPath-algebra is equivalent with Tarski's relation algebra [2] over trees. Tarski and Givant [3,5] established the link between Tarski's algebra and FO^3. Corollary 8.7 can then be used to give a new characterization, other than via pebble-games [35,37], of when two nodes in an unordered tree are indistinguishable in FO^3. In this light, connections between other fragments of the XPath algebra and finite-variable logics must be examined.

The connection between the XPath algebra and FO^3 also has ramifications with regard to complexity issues. Indeed, using a result of Grohe [38] which establishes that expression equivalence for FO^3 is decidable in polynomial time, it follows readily from Corollaries 8.13 and 8.15 that the global and local definability problems for the XPath algebra are decidable in polynomial time. Using the syntactic characterizations in this paper, one can also establish that the global and local definability problems for the other fragments of the XPath algebra are decidable in polynomial time. As mentioned in the Introduction, this feasibility suggests efficient partitioning and reduction techniques on the set of nodes and the set of paths in a document. Such techniques might be successfully applied towards various aspects of XML document processing such as indexing, access control, and document compression. This is another research direction which we are currently pursuing [11,39].

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