Linear Programming

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cs302
Spring 2012

Administrative

- Assignment 16
  - Mean: 35.6
  - Median: 36
- Assignment 17
  - Mean: 14.6
  - Median: 15
- NP-complete?

Linear programming

A linear function is a function of n variables defined by

$$f(x_1, x_2, \ldots, x_n) = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

A linear equality is a linear function with an equality constraint

$$f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$

A linear inequality is a linear function with an inequality constraint

$$f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \leq b$$
$$f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \geq b$$

Linear programming

- A linear programming problem consists of two parts
  - a linear function to maximize or minimize

$$\text{maximize } c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

  - subject to a set of linear constraints

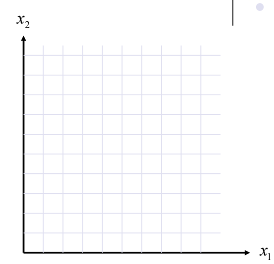
$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b$$
$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \leq b$$
$$\ldots$$
$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \geq b$$

For example

$$\begin{array}{l}\text{maximize } x_1 + x_2 \qquad \text{objective function} \\ \text{subject to} \\ \left.\begin{array}{rcr} 4x_1 - x_2 & \le & 8 \\ 2x_1 + x_2 & \le & 10 \\ 5x_1 - 2x_2 & \ge & -2 \\ x_1, x_2 & \ge & 0 \end{array}\right\} \text{constraints}\end{array}$$

For example

$$\begin{array}{l}\text{maximize } x_1 + x_2 \\ \text{subject to} \\ \begin{array}{rcr} 4x_1 - x_2 & \le & 8 \\ 2x_1 + x_2 & \le & 10 \\ 5x_1 - 2x_2 & \ge & -2 \\ x_1, x_2 & \ge & 0 \end{array}\end{array}$$



For example

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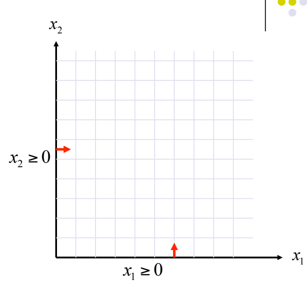

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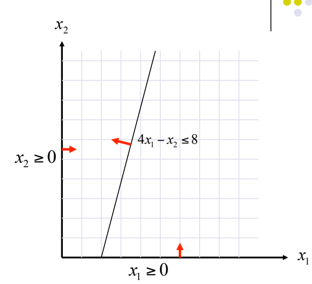


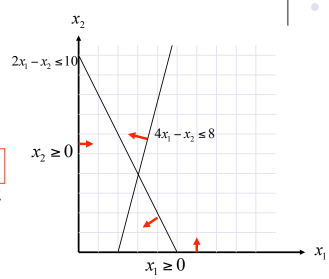
$$\boxed{x_2 \ge 4x_1 - 8}$$

For example

maximize $x_1 + x_2$

subject to

$$\begin{array}{rcr} 4x_1 - x_2 & \le & 8 \\ \boxed{2x_1 + x_2 \quad \le \quad 10} & & \\ 5x_1 - 2x_2 & \ge & -2 \\ x_1, x_2 & \ge & 0 \end{array}$$



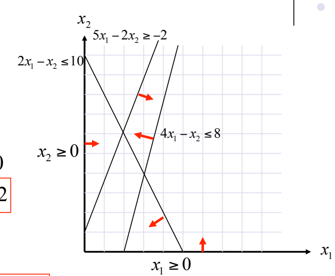
$\boxed{x_2 \le -2x_1 + 10}$

For example

maximize $x_1 + x_2$

subject to

$$\begin{array}{rcr} 4x_1 - x_2 & \le & 8 \\ 2x_1 + x_2 & \le & 10 \\ \boxed{5x_1 - 2x_2 \quad \ge \quad -2} & & \\ x_1, x_2 & \ge & 0 \end{array}$$



$\boxed{x_2 \le \frac{5}{2}x_1 + 1}$

For example

maximize $x_1 + x_2$

subject to

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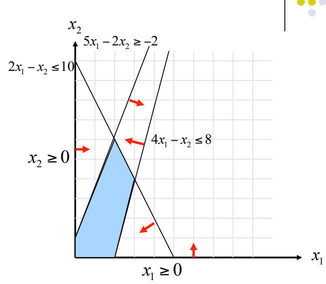


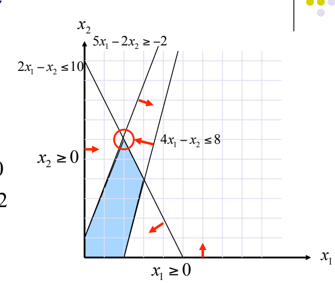
the constraints define
the **feasible region**

For example

$\boxed{\text{maximize } x_1 + x_2}$

subject to

$$\begin{array}{rcr} 4x_1 - x_2 & \le & 8 \\ 2x_1 + x_2 & \le & 10 \\ 5x_1 - 2x_2 & \ge & -2 \\ x_1, x_2 & \ge & 0 \end{array}$$



What are the values of x_i that maximize
the function within the feasible region?

Another example

- A chocolatier has two products: a basic product and a deluxe. The company makes x_1 boxes of the basic per day at a profit of \$1 each and x_2 boxes of the deluxe at a profit of \$6 each. The daily demand is 200 of the basic and 300 of the deluxe. The workforce can create 400 boxes of chocolate per day. How much of each should they create to maximize profits?

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How many variables do we need to model the problem?

Another example

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x_1 = number of boxes per day of basic

x_2 = number of boxes per day of deluxe

Another example

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What are the constraints?

Another example

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$$\begin{aligned} x_1 &\le 200 \\ x_2 &\le 300 \end{aligned}$$

Another example

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$$\begin{aligned} x_1 &\le 200 \\ x_2 &\le 300 \\ x_1 + x_2 &\le 400 \end{aligned}$$

any others?

Another example

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$$\begin{aligned} x_1 &\le 200 \\ x_2 &\le 300 \\ x_1 + x_2 &\le 400 \\ x_1, x_2 &\ge 0 \end{aligned}$$

What function are we trying to maximize/minimize?

Another example

- A chocolatier has two products: a basic product and a deluxe. The company makes x_1 boxes of the basic per day at a profit of \$1 each and x_2 boxes of the deluxe at a profit of \$6 each. The daily demand is 200 of the basic and 300 of the deluxe. The workforce can create 400 boxes of chocolate per day. How much of each should they create to maximize profits?

$$\text{maximize } x_1 + 6x_2$$

subject to

$$\begin{aligned} x_1 &\le 200 \\ x_2 &\le 300 \\ x_1 + x_2 &\le 400 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Another example

maximize $x_1 + 6x_2$

subject to

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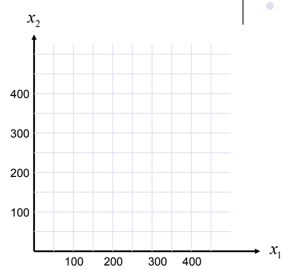

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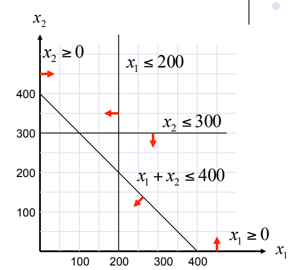

Where is the feasibility region?

Another example

maximize $x_1 + 6x_2$

subject to

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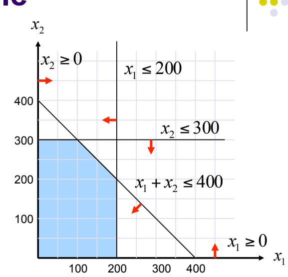

Where is the maximum within the feasibility region?

Another example

maximize $x_1 + 6x_2$

subject to

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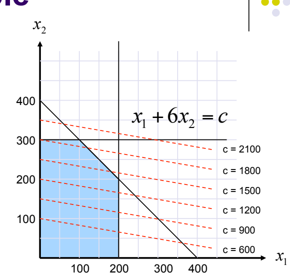

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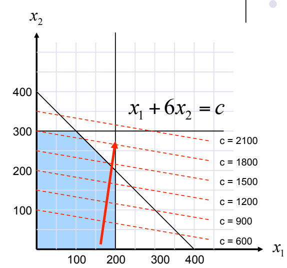
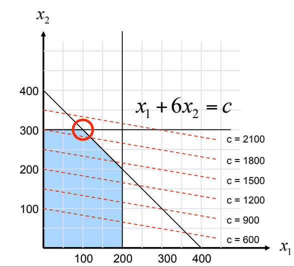


to maximize, move as far in this direction as the constraints allow

Solutions to LPs

- A solution to an LP is a vertex on the feasibility polygon
  - the very last feasible point in the direction of improving objective function



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- Except…

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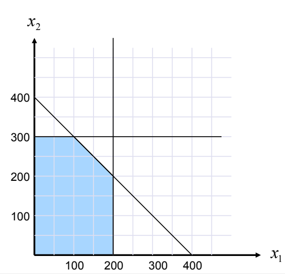

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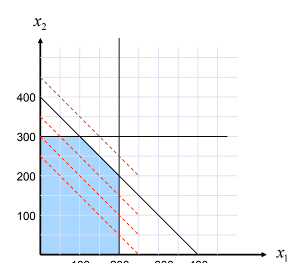

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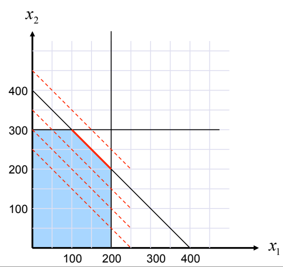

Solutions to LPs

- A solution to an LP is a vertex on the feasibility polygon
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- Except…

$$\text{maximize}\; x_1 + 6x_2$$

subject to

$$\begin{aligned} x_1 &\le 1 \\ x_1 &\ge 2 \\ &\ldots \end{aligned}$$

linear program is **infeasible**

Solutions to LPs

- A solution to an LP is a vertex on the feasibility polygon
  - the very last feasible point in the direction of improving objective function
- Except…

$$\text{maximize}\; x_1 + x_2$$

subject to

$$x_1, x_2 \ge 0$$

linear program is **unbounded**

More products

- The chocolatier decides to introduce a third product line called premium with a profit of $13. We still can only produce 400 units per day. The deluxe and premium products require the same packaging machinery. Deluxe uses one unit and premium uses 3 units. We have a total of 600 units a day of this packaging material.

$$\text{maximize}\; x_1 + 6x_2$$

subject to

$$\begin{aligned} x_1 &\le 200 \\ x_2 &\le 300 \\ x_1 + x_2 &\le 400 \\ x_1, x_2 &\ge 0 \end{aligned}$$

what changes?

More products

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Introduce a new variable x_3

$$\begin{aligned} &\text{maximize } x_1 + 6x_2 \\ &\text{subject to} \\ &\quad x_1 \le 200 \\ &\quad x_2 \le 300 \\ &\quad x_1 + x_2 \le 400 \\ &\quad \boxed{x_1, x_2, x_3 \ge 0} \end{aligned}$$

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Introduce a new constraint

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$$\begin{aligned} &\text{maximize } x_1 + 6x_2 \\ &\text{subject to} \\ &\quad x_1 \le 200 \\ &\quad x_2 \le 300 \\ &\quad \boxed{x_1 + x_2 + x_3 \le 400} \\ &\quad x_2 + 3x_3 \le 600 \\ &\quad x_1, x_2, x_3 \ge 0 \end{aligned}$$

modify existing constraints

More products

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Anything else?

More products

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modify the objective function

More products

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What does the feasibility region look like?

Feasibility region

- For n variables, the feasibility region is a polyhedron in R^n (i.e. n-dimensional space)
- Each constraint defines a R^{n-1} plane and the inequality a half-space on one side of the plane

$$\begin{array}{l} \text{maximize } x_1 + 6x_2 + 13x_3 \\ \text{subject to} \end{array}$$

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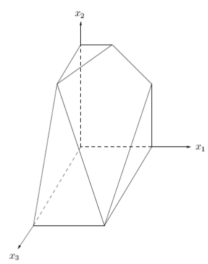

adapted from figure 7.2 of [1]

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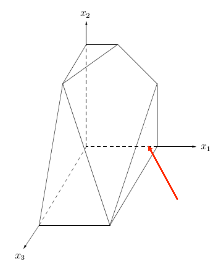

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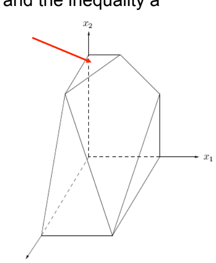

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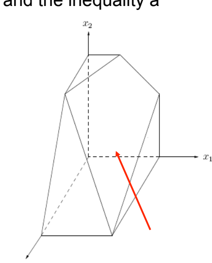

Feasibility region

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$$
\begin{array}{l}
\text{maximize } x_1 + 6x_2 + 13x_3 \\
\text{subject to}
\end{array}
$$

$$
\begin{array}{rcl}
x_1 & \le & 200 \\
x_2 & \le & 300 \\
x_1 + x_2 + x_3 & \le & 400 \\
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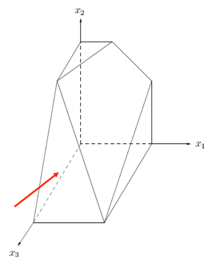
\end{array}
$$

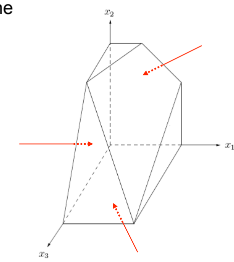


Feasibility region

- For n variables, the feasibility region is a polyhedron in R^n (i.e. n-dimensional space)
- Each constraint defines a R^{n-1} plane and the inequality a half-space on one side of the plane

$$
\begin{array}{l}
\text{maximize } x_1 + 6x_2 + 13x_3 \\
\text{subject to}
\end{array}
$$

$$
\begin{array}{rcl}
x_1 & \le & 200 \\
x_2 & \le & 300 \\
x_1 + x_2 + x_3 & \le & 400 \\
x_2 + 3x_3 & \le & 600 \\
x_1, x_2, x_3 & \ge & 0
\end{array}
$$



Vector/matrix form

$$
\begin{array}{l}
\text{maximize } c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{subject to}
\end{array}
$$

$$
\begin{array}{rcl}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & \le & b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & \le & b_2 \\
 & \ldots & \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & \le & b_m
\end{array}
$$

$$
\begin{array}{l}
\text{maximize } \mathbf{c}^T\mathbf{x} \\
\text{subject to}
\end{array}
\qquad
\begin{array}{rcl}
\mathbf{Ax} & \le & \mathbf{b} \\
\mathbf{x} & \ge & 0
\end{array}
$$

Vector/matrix form

$$
\begin{array}{l}
\text{maximize } x_1 + 6x_2 + 13x_3 \\
\text{subject to}
\end{array}
$$

$$
\begin{array}{rcl}
x_1 & \le & 200 \\
x_2 & \le & 300 \\
x_1 + x_2 + x_3 & \le & 400 \\
x_2 + 3x_3 & \le & 600
\end{array}
$$

c: [1, 6, 13]

A: [1, 0, 0

Vector/matrix form

maximize $x_1 + 6x_2 + 13x_3$
subject to

$$\begin{aligned} x_1 &\le 200 \\ \boxed{x_2 \le 300} \\ x_1 + x_2 + x_3 &\le 400 \\ x_2 + 3x_3 &\le 600 \end{aligned}$$

c: [1, 6, 13]

A: [1, 0, 0

Vector/matrix form

maximize $x_1 + 6x_2 + 13x_3$
subject to

$$\begin{aligned} x_1 &\le 200 \\ \boxed{x_2 \le 300} \\ x_1 + x_2 + x_3 &\le 400 \\ x_2 + 3x_3 &\le 600 \end{aligned}$$

c: [1, 6, 13]

A: [1, 0, 0
0, 1, 0

Vector/matrix form

maximize $x_1 + 6x_2 + 13x_3$
subject to

$$\begin{aligned} x_1 &\le 200 \\ x_2 &\le 300 \\ \boxed{x_1 + x_2 + x_3 \le 400} \\ x_2 + 3x_3 &\le 600 \end{aligned}$$

c: [1, 6, 13]

A: [1, 0, 0
0, 1, 0

Vector/matrix form

maximize $x_1 + 6x_2 + 13x_3$
subject to

$$\begin{aligned} x_1 &\le 200 \\ x_2 &\le 300 \\ \boxed{x_1 + x_2 + x_3 \le 400} \\ x_2 + 3x_3 &\le 600 \end{aligned}$$

c: [1, 6, 13]

A: [1, 0, 0
0, 1, 0
1, 1, 1

Vector/matrix form

$$\begin{aligned} &\text{maximize } x_1 + 6x_2 + 13x_3 \\ &\text{subject to} \\ &\quad x_1 \le 200 \\ &\quad x_2 \le 300 \\ &\quad x_1 + x_2 + x_3 \le 400 \\ &\quad \boxed{x_2 + 3x_3 \le 600} \end{aligned}$$

c: [1, 6, 13]

A: [1, 0, 0
0, 1, 0
1, 1, 1

Vector/matrix form

$$\begin{aligned} &\text{maximize } x_1 + 6x_2 + 13x_3 \\ &\text{subject to} \\ &\quad x_1 \le 200 \\ &\quad x_2 \le 300 \\ &\quad x_1 + x_2 + x_3 \le 400 \\ &\quad x_2 + 3x_3 \le 600 \end{aligned}$$

c: [1, 6, 13]

A: [1, 0, 0
0, 1, 0
1, 1, 1
0, 1, 3]

b: [200,
300,
400,
600]

Standard form

- An LP is in standard form if
  - It is a maximization problem
  - the constraints are all ≤

$$\mathbf{a}_i \mathbf{x} \le b_i$$

  - and all variables are non-negative, i.e. have constraints of the form

$$x_i \ge 0$$

Converting an LP into standard form

- LP is minimize instead of maximize

$$\begin{aligned} &\text{minimize } \mathbf{c}^T\mathbf{x} \\ &\text{subject to} \\ &\quad \mathbf{Ax} \le \mathbf{b} \\ &\quad \mathbf{x} \ge 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\text{maximize } -\mathbf{c}^T\mathbf{x} \\ &\text{subject to} \\ &\quad \mathbf{Ax} \le \mathbf{b} \\ &\quad \mathbf{x} \ge 0 \end{aligned}$$

Converting an LP into standard form

- LP contains constraints of the form:

$$\mathbf{a}_i\mathbf{x} \geq b_i \Rightarrow -\mathbf{a}_i\mathbf{x} \leq -b_i$$

Converting an LP into standard form

- LP contains constraints of the form:

$$\mathbf{a}_i\mathbf{x} = b_i \Rightarrow \begin{array}{l}\mathbf{a}_i\mathbf{x} \leq b_i \\ \mathbf{a}_i\mathbf{x} \geq b_i\end{array} \Rightarrow \begin{array}{l}\mathbf{a}_i\mathbf{x} \leq b_i \\ -\mathbf{a}_i\mathbf{x} \leq -b_i\end{array}$$

Converting an LP into standard form

- LP contains variable without a non-negative bound constraint

$$x_i \leq 0 \Rightarrow x'_i \geq 0$$

replace all occurrences of x_i with $-x'_i$ in the objective and constraints

Converting an LP into standard form

- LP contains variable without a non-negative bound constraint

$$x_i \geq b_i \Rightarrow x'_i \geq 0$$

substitute x'_i for x_i and change constraints and objective appropriately (i.e. $x_i - b_i$), including introducing a new variable in the objective function

Converting an LP into standard form

- LP contains variable without a non-negative bound constraint

x_i is unbounded $\Rightarrow$ $x_i^+ \geq 0$, $x_i^- \geq 0$

replace all occurrences of x_i with $x_i^+ - x_i^-$ in the objective and constraints

Solving linear programs

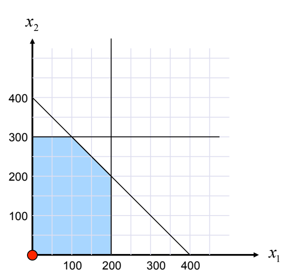
- If a solution exists, a vertex of the feasibility space is an optimal solution
- Because the objective function is linear, we can use a *hill-climbing* or greedy approach by starting at one vertex and moving to an adjacent vertex that improves the object function
- When we're on top of the "hill", i.e. we cannot move to a better neighboring vertex, we' re done
- The simplex method does exactly this

Simplex method

Start: [0,0]

Objective score:

$$
\begin{aligned}
&\text{maximize } x_1 + x_2 \\
&\text{subject to} \\
&\quad x_1 \leq 200 \\
&\quad x_2 \leq 300 \\
&\quad x_1 + x_2 \leq 400 \\
&\quad x_1, x_2 \geq 0
\end{aligned}
$$

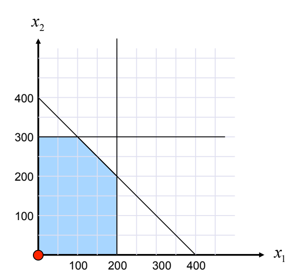


Simplex method

Start: [0,0]

Objective score: 0

$$
\begin{aligned}
&\text{maximize } x_1 + x_2 \\
&\text{subject to} \\
&\quad x_1 \leq 200 \\
&\quad x_2 \leq 300 \\
&\quad x_1 + x_2 \leq 400 \\
&\quad x_1, x_2 \geq 0
\end{aligned}
$$



Simplex method

Adjacent vertices

[0, 300]

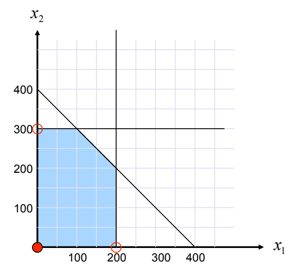
Objective score:

[200, 0]

Objective score:

$$\begin{array}{l} \text{maximize}\, x_1 + x_2 \\ \text{subject to} \end{array}$$

$$\begin{array}{rcl} x_1 & \le & 200 \\ x_2 & \le & 300 \\ x_1 + x_2 & \le & 400 \\ x_1, x_2 & \ge & 0 \end{array}$$



Simplex method

Adjacent vertices

[0, 300]

Objective score: 300

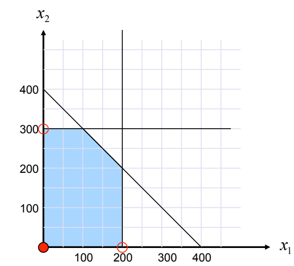
[200, 0]

Objective score: 200

either option improves the score

$$\begin{array}{l} \text{maximize}\, x_1 + x_2 \\ \text{subject to} \end{array}$$

$$\begin{array}{rcl} x_1 & \le & 200 \\ x_2 & \le & 300 \\ x_1 + x_2 & \le & 400 \\ x_1, x_2 & \ge & 0 \end{array}$$



Simplex method

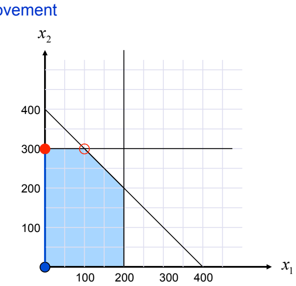
Adjacent vertices

[100, 300]

Objective score: 400 improvement

$$\begin{array}{l} \text{maximize}\, x_1 + x_2 \\ \text{subject to} \end{array}$$

$$\begin{array}{rcl} x_1 & \le & 200 \\ x_2 & \le & 300 \\ x_1 + x_2 & \le & 400 \\ x_1, x_2 & \ge & 0 \end{array}$$



Simplex method

Adjacent vertices

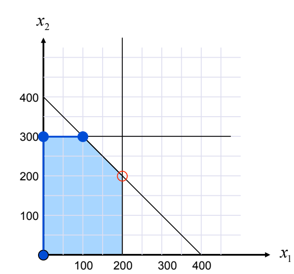
[200, 200]

Objective score: 400

No improvement, so we're done

$$\begin{array}{l} \text{maximize}\, x_1 + x_2 \\ \text{subject to} \end{array}$$

$$\begin{array}{rcl} x_1 & \le & 200 \\ x_2 & \le & 300 \\ x_1 + x_2 & \le & 400 \\ x_1, x_2 & \ge & 0 \end{array}$$

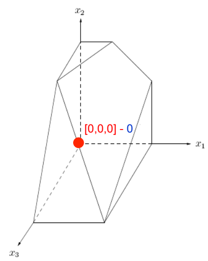


Another simplex example

Start at [0,0,0]

Adjacent vertices

$$\text{maximize } x_1 + 6x_2 + 13x_3$$
subject to
$$\begin{array}{rcl} x_1 & \le & 200 \\ x_2 & \le & 300 \\ x_1 + x_2 + x_3 & \le & 400 \\ x_2 + 3x_3 & \le & 600 \\ x_1, x_2, x_3 & \ge & 0 \end{array}$$

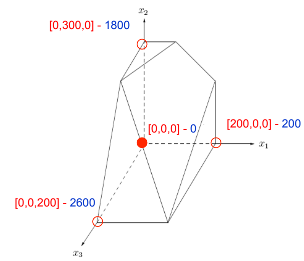


Another simplex example

Start at [0,0,0]

Adjacent vertices

$$\text{maximize } x_1 + 6x_2 + 13x_3$$
subject to
$$\begin{array}{rcl} x_1 & \le & 200 \\ x_2 & \le & 300 \\ x_1 + x_2 + x_3 & \le & 400 \\ x_2 + 3x_3 & \le & 600 \\ x_1, x_2, x_3 & \ge & 0 \end{array}$$

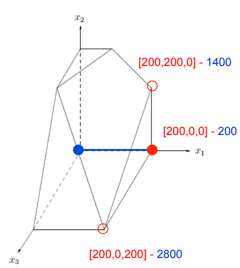


Another simplex example

[200,0,0]

Adjacent vertices

$$\text{maximize } x_1 + 6x_2 + 13x_3$$
subject to
$$\begin{array}{rcl} x_1 & \le & 200 \\ x_2 & \le & 300 \\ x_1 + x_2 + x_3 & \le & 400 \\ x_2 + 3x_3 & \le & 600 \\ x_1, x_2, x_3 & \ge & 0 \end{array}$$

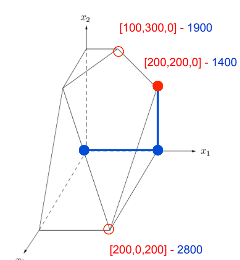


Another simplex example

[200,200,0]

Adjacent vertices

$$\text{maximize } x_1 + 6x_2 + 13x_3$$
subject to
$$\begin{array}{rcl} x_1 & \le & 200 \\ x_2 & \le & 300 \\ x_1 + x_2 + x_3 & \le & 400 \\ x_2 + 3x_3 & \le & 600 \\ x_1, x_2, x_3 & \ge & 0 \end{array}$$

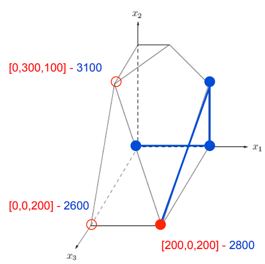


Another simplex example

[200,0,200]

Adjacent vertices

$$\begin{array}{rcl} \text{maximize } x_1 + 6x_2 + 13x_3 & & \\ \text{subject to} & & \\ x_1 & \le & 200 \\ x_2 & \le & 300 \\ x_1 + x_2 + x_3 & \le & 400 \\ x_2 + 3x_3 & \le & 600 \\ x_1, x_2, x_3 & \ge & 0 \end{array}$$



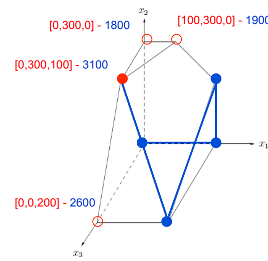
Another simplex example

[0,300,100]

Adjacent vertices

All others are lower, optimum found

$$\begin{array}{rcl} \text{maximize } x_1 + 6x_2 + 13x_3 & & \\ \text{subject to} & & \\ x_1 & \le & 200 \\ x_2 & \le & 300 \\ x_1 + x_2 + x_3 & \le & 400 \\ x_2 + 3x_3 & \le & 600 \\ x_1, x_2, x_3 & \ge & 0 \end{array}$$



Simplex algorithm

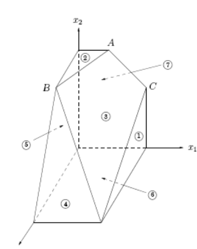
let *v* be any vertex in the feasibility region

While there is a neighbor *v'* of *v* with better objective value
set *v* = *v'*

- 3 things to work out:
  - determine a starting vertex
  - check whether current vertex is optimal
  - determine where to move next

What defines a vertex?

A vertex is specified by a set of *n* inequalities
(i.e. at the intersection of *n* hyperplanes)

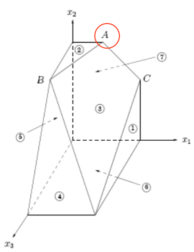


$$\begin{array}{rl} \max & x_1 + 6x_2 + 13x_3 \\ & x_1 \le 200 \quad (1) \\ & x_2 \le 300 \quad (2) \\ & x_1 + x_2 + x_3 \le 400 \quad (3) \\ & x_2 + 3x_3 \le 600 \quad (4) \\ & x_1 \ge 0 \quad (5) \\ & x_2 \ge 0 \quad (6) \\ & x_3 \ge 0 \quad (7) \end{array}$$

adapted from figure 7.12 of [1]

What defines a vertex?

A vertex is specified by a set of *n* inequalities
(i.e. at the intersection of *n* hyperplanes)

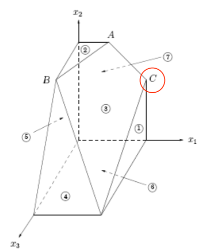


$$\begin{array}{rr} \max & x_1 + 6x_2 + 13x_3 \\ & x_1 \le 200 \quad ① \\ & \boxed{x_2 \le 300 \quad ②} \\ & \boxed{x_1 + x_2 + x_3 \le 400 \quad ③} \\ & x_2 + 3x_3 \le 600 \quad ④ \\ & x_1 \ge 0 \quad ⑤ \\ & x_2 \ge 0 \quad ⑥ \\ & \boxed{x_3 \ge 0 \quad ⑦} \end{array}$$

adapted from figure 7.12 of [1]

What defines a vertex?

A vertex is specified by a set of *n* inequalities
(i.e. at the intersection of *n* hyperplanes)

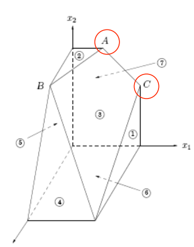


$$\begin{array}{rr} \max & x_1 + 6x_2 + 13x_3 \\ & \boxed{x_1 \le 200 \quad ①} \\ & x_2 \le 300 \quad ② \\ & \boxed{x_1 + x_2 + x_3 \le 400 \quad ③} \\ & x_2 + 3x_3 \le 600 \quad ④ \\ & x_1 \ge 0 \quad ⑤ \\ & x_2 \ge 0 \quad ⑥ \\ & \boxed{x_3 \ge 0 \quad ⑦} \end{array}$$

adapted from figure 7.12 of [1]

Can we define neighbors with respect to constraints?

A vertex is specified by a set of *n* inequalities
(i.e. at the intersection of *n* hyperplanes)



$$\begin{array}{rr} \max & x_1 + 6x_2 + 13x_3 \\ & x_1 \le 200 \quad ① \\ & x_2 \le 300 \quad ② \\ & \boxed{x_1 + x_2 + x_3 \le 400 \quad ③} \\ & x_2 + 3x_3 \le 600 \quad ④ \\ & x_1 \ge 0 \quad ⑤ \\ & x_2 \ge 0 \quad ⑥ \\ & \boxed{x_3 \ge 0 \quad ⑦} \end{array}$$

adapted from figure 7.12 of [1]

Determining the starting vertex

We'll assume that the origin is in the feasible region and start here

$$\begin{array}{l} \text{maximize } \mathbf{c}^T \mathbf{x} \\ \text{subject to} \\ \qquad \mathbf{Ax} \le \mathbf{b} \\ \qquad \boxed{\mathbf{x} \ge 0} \end{array}$$

Is it always feasible?

Is the origin the optimal solution?

- Assuming the origin is feasible
- When is the origin an optimal vertex?
  - if all $c_i \leq 0$
  - Why?

$$\begin{aligned} &\text{maximize } \mathbf{c}^T\mathbf{x} \\ &\text{subject to} \\ &\quad \mathbf{Ax} \leq \mathbf{b} \\ &\quad \mathbf{x} \geq 0 \end{aligned}$$

Determining where to move next

- If some $c_i > 0$ then the origin is not optimal
  - We can possibly increase the value of x_i and it will increase the objective function
- How much can we increase x_i?
  - Until we hit another constraint (i.e. a vertex)

$$\begin{aligned} &\text{maximize } \mathbf{c}^T\mathbf{x} \\ &\text{subject to} \\ &\quad \mathbf{Ax} \leq \mathbf{b} \\ &\quad \mathbf{x} \geq 0 \end{aligned}$$

For example

$$\begin{aligned} &\text{maximize } 2x_1 + 5x_2 \\ &\text{subject to} \\ &\quad 2x_1 - x_2 \leq 4 \\ &\quad x_1 + 2x_2 \leq 9 \\ &\quad -x_1 + x_2 \leq 3 \\ &\quad x_1 \geq 0 \\ &\quad x_2 \geq 0 \end{aligned}$$

Is [0, 0] a feasible vertex?

Yes.

We have $\mathbf{x} \geq 0$ and all b_i are positive

For example

$$\begin{aligned} &\text{maximize } 2x_1 + 5x_2 \\ &\text{subject to} \\ &\quad 2x_1 - x_2 \leq 4 \\ &\quad x_1 + 2x_2 \leq 9 \\ &\quad -x_1 + x_2 \leq 3 \\ &\quad x_1 \geq 0 \\ &\quad x_2 \geq 0 \end{aligned}$$

Is it optimal?

no, both c_i are positive

For example

$$\begin{array}{lcl} \text{maximize } 2x_1 + 5x_2 & & \\ \text{subject to} & & \\ 2x_1 - x_2 & \leq & 4 \\ x_1 + 2x_2 & \leq & 9 \\ \boxed{-x_1 + x_2 \leq 3} & & \\ x_1 & \geq & 0 \\ x_2 & \geq & 0 \end{array}$$

How far can we increase x_2?

$x_2 = 3$

This gives us a new vertex at [0, 3]

Simplex algorithm continued

- We've shown how to check for optimality and move to a new vertex when we are at the origin
- After we've moved to a new vertex, now what?
- Idea: we can transform the problem so that the new point is at the origin
- The details are beyond the scope of this class, but it's not too hard…

Simplex algorithm

let v be any vertex in the feasibility region

transform the LP so that v is the origin

while a c_i exists such that $c_i > 0$ (i.e. a better solution/vertex exists)

- pick some such c_i and increase x_i until a constraint is *tight* (i.e. increasing x_i further would move out of the feasibility space
- set v = v'
- transform the LP so that v is the origin

Running time

- let v be any vertex in the feasibility region
- transform the LP so that v is the origin
- while a c_i exists such that $c_i > 0$ (i.e. a better solution/vertex exists)
  - pick some such c_i and increase x_i until a constraint is *tight* (i.e. increasing x_i further would move out of the feasibility space
  - set v = v'
  - transform the LP so that v is the origin

Given an LP, we have n variables, m inequality constraints and n constraints of the form $x \geq 0$

Running time

- let v be any vertex in the feasibility region
- transform the LP so that v is the origin
- while a c_i exists such that $c_i > 0$ (i.e. a better solution/vertex exists)
  - pick some such c_i and increase x_i until a constraint is *tight* (i.e. increasing x_i further would move out of the feasibility space
  - set v = v'
  - transform the LP so that v is the origin

What is the per iteration cost?

Running time

- let v be any vertex in the feasibility region
- transform the LP so that v is the origin
- while a c_i exists such that $c_i > 0$ (i.e. a better solution/vertex exists)
  - pick some such c_i and increase x_i until a constraint is *tight* (i.e. increasing x_i further would move out of the feasibility space
  - set v = v'
  - transform the LP so that v is the origin

O(nm)

m constraints with up to n variables to determine how far we can move c_i

For each constraint:

Solve: $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n$ for all x fixed but x_i

Running time

- let v be any vertex in the feasibility region
- transform the LP so that v is the origin
- while a c_i exists such that $c_i > 0$ (i.e. a better solution/vertex exists)
  - pick some such c_i and increase x_i until a constraint is *tight* (i.e. increasing x_i further would move out of the feasibility space
  - set v = v'
  - transform the LP so that v is the origin

O(nm)

Running time

- let v be any vertex in the feasibility region
- transform the LP so that v is the origin
- while a c_i exists such that $c_i > 0$ (i.e. a better solution/vertex exists)
  - pick some such c_i and increase x_i until a constraint is *tight* (i.e. increasing x_i further would move out of the feasibility space
  - set v = v'
  - transform the LP so that v is the origin

How many iterations?

Running time

- let v be any vertex in the feasibility region
- transform the LP so that v is the origin
- while a c_i exists such that $c_i > 0$ (i.e. a better solution/vertex exists)
  - pick some such c_i and increase x_i until a constraint is *tight* (i.e. increasing x_i further would move out of the feasibility space
  - set v = v'
  - transform the LP so that v is the origin

O(vertices)

How many vertices?

Each vertex is defined by n constraints and we have m + n constraints: $\binom{n+m}{n}$

exponential!

Running time of simplex

- Worst case is exponential
- There are theoretical examples that can be created where the running time is exponential
- In practice, the algorithm is much faster

Other LP solvers

- Many other LP algorithms exist
  - ellipsoid algorithm
  - interior point methods (many of these)
- Both of the above algorithms are O(polynomial time)
- State of the art run-times are a toss up between interior point methods and simplex solvers

Another example

- Carpet company: From our analysts, we're given carpet demand estimates for the next calendar year: $d_1, d_2, \ldots, d_n$. The company has 30 employees, each of which makes 20 carpets per month and gets paid $2000/mo.
- The demand fluctuates from month to month. We can handle this demand in three ways:
  - Overtime: Overtime costs 80% more than regular pay. Workers can put in at most 30% overtime
  - Hiring and firing, at a cost of $320 and $400 respectively per worker
  - Store extra carpets at a cost of $8 per carpet per month. We must end the year without any stored carpets.

What is the profit for a month?

- Carpet company: From our analysts, we're given carpet demand estimates for the next calendar year: $d_1, d_2, \ldots, d_n$. The company has 30 employees, each of which makes 20 carpets per month and gets paid $2000/mo.
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profit = revenue - cost

revenue = carpets_sold * price

cost = employees * 2000 +
overtime * 180 +
hirings * 320 +
firings * 400 +
storage * 8

Defining variables

- Carpet company: From our analysts, we're given carpet demand estimates for the next calendar year: $d_1, d_2, \ldots, d_n$. The company has 30 employees, each of which makes 20 carpets per month and gets paid $2000/mo.
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c_i = carpets sold in month i
x_i = carpets made in month i
w_i = number of employees during month i
o_i = number of carpets made with overtime
h_i = number of hirings at beginning of month i
f_i = number of firings at beginning of month i
s_i = number of carpets stored in month i

Constraints

- Carpet company: From our analysts, we're given carpet demand estimates for the next calendar year: $d_1, d_2, \ldots, d_n$. The company has 30 employees, each of which makes 20 carpets per month and gets paid $2000/mo.
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Constraints on carpets sold?

$$c_i \le d_i$$

$$c_i \le x_i + s_i$$

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How many carpets are made a month?

$$x_i = 20w_i + o_i$$

Constraints

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How many workers do we have each month?

$$w_i = w_{i-1} + h_i - f_i$$

Constraints

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f_i = number of firings at beginning of month i

s_i = number of carpets stored in month i

How many carpets are stored?

$$s_i = s_{i-1} + x_i - c_i$$

Constraints

- Carpet company: From our analysts, we' re given carpet demand estimates for the next calendar year: $d_1, d_2, \ldots, d_n$. The company has 30 employees, each of which makes 20 carpets per month and gets paid \$2000/mo.
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$$o_i \le 6w_i$$

Constraints

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s_i = number of carpets stored in month i

Any other constraints?

$$c_i, x_i, w_i, o_i, h_i, f_i, s_i \ge 0$$

Maximizing profit

profit = revenue - cost

revenue = carpets_sold * price

cost = employees * 2000 +
overtime * 180 +
hirings * 320 +
firings * 400 +
storage * 8

Revenue?

$$revenue = c_i price_i$$

Maximizing profit

profit = revenue - cost

revenue = carpets_sold * price

cost = employees * 2000 +
overtime * 180 +
hirings * 320 +
firings * 400 +
storage * 8

Cost?

$$2000\sum_i w_i \quad 180\sum_i o_i \quad 320\sum_i h_i \quad 400\sum_i f_i \quad 8\sum_i f_i$$

Objective function

profit = revenue - cost

revenue = carpets_sold * price

cost = employees * 2000 +
overtime * 180 +
hirings * 320 +
firings * 400 +
storage * 8

$$\text{maximize}\, c_i price_i - (2000\sum_i w_i + 180\sum_i o_i + 320\sum_i h_i + 400\sum_i f_i + 8\sum_i f_i)$$

The solution

- Given our problem formulation, we can plug it into a solver (or write our own) and we' ll get a solution back if one exists
- Are we done?
- Our solution may have non-integer values. How do we sell .3 of a carpet?
  - If the variable values are large, we can round and we won't be too far away from optimal
  - If the variable values are small, then we need to be more careful about our rounding decisions
- Integer (linear) programming is the linear programming problem with integer solutions
  - Like the 0-1 knapsack problem vs. the fractional knapsack problem, the fractional problem is easier
  - Integer programming is NP-complete

Uses

- operational problems
- network flow
- planning
- microeconomics

Beyond LPs

$$\text{maximize } f_0(\mathbf{x})$$
$$\text{subject to}$$
$$f_i(\mathbf{x}) \le b_i$$

- Convex optimization
  - $f_i(c_1x + c_2y) = c_1f_i(x) + c_2f_i(y)$

References

- [1] Algorithms (2008). Sanjoy Dasgupta, Christos Papadimitiou and Umesh Vazirani.